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PARTON DISTRIBUTIONS IN RADIATIVE CORRECTIONS TO THE CROSS SECTION OF ELECTRON-PROTON SCATTERING

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Parton distributions in radiative corrections to the cross section of electron-proton scattering

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Abstract

The structure function approach and the parton picture, developed for the theoretical description of the deep inelastic electron-proton scattering, also proved to be very effective for calculation of radiative corrections in Quantum Electrodynamics. We use them to calculate radiative corrections to the cross section of electron-proton scattering due to electron-photon interaction, in the experimental setup with the recoil proton detection, proposed by A.A. Vorobev to measure the proton radius. In the one-loop approximation, explicit expressions for these corrections are obtained for arbitrary momentum transfers. It is shown that, at momentum transfers small compared with the proton mass, various contributions to the corrections mutually cancel each other with power accuracy. In two loops, the corrections are obtained in the leading logarithmic approximation.

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1 Introduction

Although after appearance of the paper [1] the "proton radius puzzle" [2, 3] – the striking difference in the proton radius values extracted from the 2S-2P transition in muonic hydrogen [4, 5] and obtained from electron-proton scattering and hydrogen spectroscopy [6] (for a review, see Ref. [7]) – seems partially (regarding the contradiction between the results of experiments with muonic and usual hydrogen) resolved, the contradiction between muonic hydrogen and electron-proton scattering results remains. Moreover, latest electron scattering experiments at Jlab [8] and MAMI [9] and hydrogen spectroscopy experiments [10, 11] not only did not resolve the puzzle, but made it even more confusing.

Currently new scattering experiments are being prepared. A distinctive feature of one of them [12], which was suggested by A.A. Vorobev and has to be performed with a low-intensity electron beam at MAMI, is that instead of detecting a scattered electron, as in previous experiments, it is supposed to detect with a high precision a recoil proton in the region of low momentum transfers ($0.04\ \mathrm{GeV}^2 > Q^2 > 0.001\ \mathrm{GeV}^2$). The aim is to extract the proton radius with 0.6 percent precision, which could be decisive in solving the proton radius puzzle. To this end, it is planned to achieve 0.2 percent accuracy of the cross section $d\sigma/(dQ^2)$ measurement.

Such accuracy requires precise account of radiative corrections. Although calculation of the radiative corrections to the electron-proton scattering cross section has a long history (see, for example, Refs. [13]-[16][1], and recent reviews [18]-[20]) the results obtained before cannot be completely applied to the experiment discussed above. The reason it that they were obtained for experiments in which scattered electrons were detected (honestly speaking, there was the experiment [21] where the recoil proton was detected; but calculation of the radiative corrections to this experiment was not explained). Since the radiative corrections include contributions of inelastic processes with photon emission, they depend strongly on experimental conditions, so that the corrections calculated for experiments with detection of scattered electrons are not suitable for experiments with detection of recoil proton. It occurs [22] that the radiative corrections for experiments with detection of recoil proton have a new unexpected and pleasant property – cancellation of the most important corrections, which are due to electron-photon interaction[2], in the region of low momentum transfers. In [22], the cancellation of not only infrared, but also collinear singularities was shown and a simple physical explanation of this phenomenon was given. It was also argued that in the one-loop approximation the accuracy of the cancellation is higher than the logarithmic, and the terms not having the collinear singularities (constant terms) are cancelled as well.

Here we refine the results of [22] and get new ones, with a wider scope of applicability, using the structure function approach and the parton picture, developed for the theoretical description of the deep inelastic electron-proton scattering [25]-[29] and adopted in [30] for calculation of radiative corrections in QED.

[1]Higher order corrections to the lepton line was considered for the standard experimental set-up with scattered electron measurement in [17].

[2]Cancellation of leptonic radiative corrections to deep inelastic scattering was discussed in [23] and[24].

2 Statement of the approach

Following [22], we denote four-momenta of initial and final electron (proton) as l (p) and l' (p'); $l^2 = l'^2 = m^2$, $p^2 = p'^2 = M^2$, and use the designations $Q^2 = -q^2$, $q = p - p'$ both for elastic the and inelastic processes.

The cross section of electron-proton scattering with radiative corrections due only to electron interaction can be considered as inclusive proton-electron scattering cross section. It means that it can be written as

$$E'_p \frac{d^3\sigma}{d^3p'} = \frac{(\alpha(Q^2))^2}{Q^4} \frac{1}{\sqrt{(pl)^2 - m^2M^2}} J^{\mu\nu}(p,p') W_{\mu\nu}(l,q) \ , \tag{2.1}$$

where

$$\alpha(Q^2) = \frac{\alpha}{1 - \mathcal{P}(q^2)} \ , \tag{2.2}$$

$\mathcal{P}(q^2)$ is the vacuum polarisation, which is real at $q^2 = -Q^2 < 0$; $J^{\mu\nu}(p,p')$ is the proton current tensor

$$J^{\mu\nu}(p,p') = \overline{\sum}_{pol} J^\mu J^{*\nu} \ , \tag{2.3}$$

$\overline{\sum}_{pol}$ means summation over final polarisations and averaging over initial ones,

$$J^\mu = \bar{u}(p') \left(f_1(Q^2)\, \gamma^\mu + f_2(Q^2)\, \frac{[\gamma^\mu, \gamma^\nu] q_\nu}{4M} \right) u(p) \ , \tag{2.4}$$

$f_1(Q^2)$ and $f_2(Q^2)$ are the the Dirac and Pauli form factors of the proton, and $W_{\mu\nu}(l,q)$ is the deep inelastic scattering tensor,

$$W_{\mu\nu}(l,q) = \frac{1}{4\pi} \overline{\sum}_X \langle l | j^{(e)}_\nu(0) | X \rangle \ \langle X | j^{(e)}_\mu(0) | l \rangle \ (2\pi)^4 \delta(q + l - p_X) \ . \tag{2.5}$$

Here $|l\rangle$ is the initial electron state, $|X\rangle$ is any state which can be produced in photon-electron collisions, $\overline{\sum}_X$ means averaging over initial electron polarisations and summation over discrete and integration over continuous variables of $|X\rangle$, $j^{(e)}_\mu$ is the electron electromagnetic current operator.

Taking into account conservation of the current, one can represent $W_{\mu\nu}$ in the form

$$W^{\mu\nu}(l,q) = F_1(x,Q^2) \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{F_2(x,Q^2)}{(lq)} \left(l^\mu - \frac{(lq)}{q^2} q^\mu \right) \left(l^\nu - \frac{(lq)}{q^2} q^\nu \right) \ , \tag{2.6}$$

where

$$x = Q^2/(2(lq)) \tag{2.7}$$

is the Bjorken variable and $F_i(x,Q^2)$ are the electron structure functions. They are expressed in terms of the convolutions W_i of the tensor $W^{\mu\nu}(l,q)$

$$W_g = W_{\mu\nu}(l,q) g^{\mu\nu} \ , \qquad W_l = W_{\mu\nu}(l,q) l^\mu l^\nu \tag{2.8}$$

using the relations

$$F_1(x,Q^2) = \frac{1}{2} \left(\frac{Q^2\, W_l}{Q^2 m^2 + (ql)^2} - W_g \right) \ , \tag{2.9}$$

$$F_2(x,Q^2) = \frac{1}{2}\frac{Q^2(ql)}{(Q^2m^2+(ql)^2)}\left(\frac{3Q^2\,W_l}{Q^2m^2+(ql)^2} - W_g\right) \ . \tag{2.10}$$

Calculating the tensor $J^{\mu\nu}$,

$$J^{\mu\nu} = G_M^2(Q^2)\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) + \frac{Q^2G_M^2(Q^2)+4M^2G_E^2(Q^2)}{4M^2+Q^2}P^\mu P^\nu \ , \tag{2.11}$$

where $P = p+p'$, $G_E(Q^2)$ and $G_M(Q^2)$ are the proton electric and magnetic form factors,

$$G_M(Q^2) = f_1(Q^2)+f_2(Q^2),\ \ G_E(Q^2) = f_1(Q^2) - \frac{Q^2}{4M^2}f_2(Q^2) \ , \tag{2.12}$$

performing tensor convolution and using

$$\frac{d^3p'}{2E'_p} = \frac{\pi}{4}\frac{Q^2dQ^2dx}{x^2\sqrt{(pl)^2-m^2M^2}} \ , \tag{2.13}$$

we obtain

$$\begin{aligned}\frac{d\sigma}{dQ^2dx} = \frac{\pi(\alpha(Q^2))^2}{2x^2Q^2((pl)^2-m^2M^2)}\Big[(2Q^2G_M^2 - 4M^2G_E^2)F_1(x,Q^2)+ \\ \left(-G_M^2(m^2Q^2+(ql)^2) + \frac{Q^2G_M^2+4M^2G_E^2}{4M^2+Q^2}(Pl)^2\right)\frac{F_2(x,Q^2)}{(ql)}\Big] \ .\end{aligned} \tag{2.14}$$

Here

$$(ql) = \frac{Q^2}{2x} \ , \qquad (Pl) = 2(pl) - \frac{Q^2}{2x} \ , \qquad (pl) = ME_l \ , \tag{2.15}$$

where E_l is the energy of the incident electron in the rest frame of the initial proton.

In the Born approximation, the cross section $\frac{d\sigma_B}{dQ^2}$ is determined by (2.14) with $\alpha(Q^2) = \alpha,\ F_2(x,Q^2) = 2F_1(x,Q^2) = \delta(1-x)$:

$$\frac{d\sigma_B}{dQ^2} = \frac{\pi\alpha^2M^2}{Q^4}\frac{(4(pl)-Q^2)^2+(Q^2+4M^2)(Q^2-4m^2)}{(Q^2+4M^2)((pl)^2-m^2M^2)}(\epsilon G_E^2+\tau G_M^2) \ , \tag{2.16}$$

$$\epsilon = \frac{(4(pl)-Q^2)^2 - Q^2(Q^2+4M^2)}{(4(pl)-Q^2)^2+(Q^2+4M^2)(Q^2-4m^2)} \ , \ \ \tau = \frac{Q^2}{4M^2} \ . \tag{2.17}$$

Formula (2.14) gives the exact expression for the cross section of electron-proton scattering taking into account all processes of electron-photon interaction. The radiation correction due to this interaction is determined by the equation

$$\delta_{e\gamma} = \frac{\int_0^1 dx\ \frac{d\sigma}{dQ^2dx}}{\frac{d\sigma_B}{dQ^2}} - 1 \tag{2.18}$$

and can be written as

$$\delta_{e\gamma} = \frac{1+\delta^e}{(1-\mathcal{P}(q^2))^2} - 1 \ , \tag{2.19}$$

where δ^e is the correction associated with the electron structure, that is, with the difference $F_2(x, Q^2)$ and $2F_1(x, Q^2)$ from $\delta(1-x)$.

Our main goal here is to calculate just this correction. As for he vacuum polarisation $\mathcal{P}(q^2)$, it is well known and we have nothing new to say about it. For completeness, we provide the necessary information in Appendix A.

In the proposed experiments to measure the proton radius, the momentum transfers are large compared to the electron mass, $Q^2 \gg m^2$. Below, we will be mainly interested in this particular area. Here, it is convenient to use the following representation of the cross sections (2.14) and (2.16)

$$\begin{aligned}\frac{d\sigma}{dQ^2dx} = \frac{4\pi(\alpha(Q^2))^2}{Q^4}\Bigg[&\frac{F_2(x,Q^2)}{x}R(x,Q^2)\\ &+\frac{Q^2(2Q^2G_M^2-4M^2G_E^2)}{8x^2(pl)^2}\left(F_1(x,Q^2)-\frac{F_2(x,Q^2)}{2x}\right)\Bigg] \ ,\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}R(x,Q^2) = &\left(1-\frac{Q^2}{2x(lp)}\right)\frac{Q^2G_M^2+4M^2G_E^2}{4M^2+Q^2}\\ &+\frac{Q^2}{8x^2(pl)^2}\frac{Q^2(2M^2+Q^2)G_M^2-8M^4G_E^2}{4M^2+Q^2} \ ,\end{aligned} \tag{2.21}$$

and

$$\frac{d\sigma_B}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}R(1,Q^2) \ . \tag{2.22}$$

3 Elastic scattering

For the elastic scattering, when $|X\rangle$ in (2.5) are the one-electron states with the momentum l', we have

$$\langle X|j_\mu^{(e)}(0)|l\rangle = \langle l'|j_\mu^{(e)}(0)|l\rangle = \bar{u}(l')\left(f_1^e(Q^2)\,\gamma_\mu - f_2^e(Q^2)\,\frac{[\gamma_\mu,\gamma^\nu]q_\nu}{4m}\right)u(l) \ , \tag{3.1}$$

where $f_i^e(Q^2)$ are the electron form factors. Using (2.9), (2.10) and (2.8), one obtains for the elastic contributions F_i^{el} to the electron structure functions F_i

$$\begin{aligned}F_1^{el}(x,Q^2) &= \frac{1}{2}\delta(1-x)\big(f_1^e(Q^2)+f_2^e(Q^2)\big)^2 \ ,\\ F_2^{el}(x,Q^2) &= \delta(1-x)\Big[\big(f_1^e(Q^2)\big)^2+\frac{Q^2}{4m^2}\big(f_2^e(Q^2)\big)^2\Big] \ .\end{aligned} \tag{3.2}$$

Eq. (2.14) then gives

$$\begin{aligned}\frac{d\sigma^{el}}{dQ^2} = \frac{\pi(\alpha(Q^2))^2}{4Q^2((pl)^2-m^2M^2)}\Big[&(4(pl)-Q^2)^2\frac{(Q^2G_M^2+4M^2G_E^2)(Q^2g_M^2+4m^2g_E^2)}{Q^2(Q^2+4M^2)(Q^2+4m^2)}\\ &+\big(Q^2G_M^2-4M^2G_E^2\big)g_M^2-4m^2G_M^2g_E^2\Big] \ ,\end{aligned} \tag{3.3}$$

where

$$g_M = f_1^e(Q^2) + f_2^e(Q^2), \qquad\qquad g_E = f_1^e(Q^2) - \frac{Q^2}{4m^2} f_2^e(Q^2) \ . \tag{3.4}$$

Formally, Eq. (3.3) gives the exact expression for the cross section of elastic electron-proton scattering with one-photon exchange. But essentially it has no physical meaning due to the infrared singularity. Taking into account all terms of the expansion in terms of the coupling constant α makes it zero, and each term of the expansion requires the regularization of this singularity. If the infrared divergency is regularised by the photon mass λ, in the one-loop approximation one has [31]

$$\begin{aligned} f_1^e(Q^2) = 1 - \frac{\alpha}{\pi\beta}\Big[\big(\ln\xi - \beta\big)\big(\ln\frac{m}{\lambda} - 1\big) - \frac{1}{4}\ln^2\xi + \ln\xi\ln(1+\xi) + \frac{\pi^2}{12} \\ + \mathrm{Li}_2(-\xi) + \frac{\beta(\xi-1)}{\xi+1}\ln\xi \Big] \ , \end{aligned} \tag{3.5}$$

$$f_2^e(Q^2) = \frac{\alpha}{2\pi}\frac{\sqrt{1-\beta^2}}{\beta}\ln\xi \ , \tag{3.6}$$

where β is the velocity of one of the electrons in the rest frame of the other,

$$\beta = \frac{\sqrt{Q^2(Q^2+4m^2)}}{Q^2+2m^2} \ , \qquad \xi = \sqrt{\frac{1+\beta}{1-\beta}} \ , \qquad \mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t}\ln(1-t) \ . \tag{3.7}$$

The part δ^e_{vertex} of the associated with the electron structure correction δ^e introduced by elastic scattering is determined by the difference between the vertex (3.1) and the Born one, in which $f_1^e(Q^2) = 1$ and $f_2^e(Q^2) = 0$. At $Q^2 \gg m^2$ the Pauli form factor $f_2^e(Q^2)$ is suppressed by a power-law, $f_2^e(Q^2) \sim m^2/Q^2$, so that in the one-loop approximation we have

$$\begin{aligned} \delta^e_{vertex} &= 2\big(f_1^e(Q^2) - 1\big) \\ &= \frac{\alpha}{\pi}\left[-\left(\ln\frac{Q^2}{m^2} - 1\right)\ln\frac{m^2}{\lambda^2} - \frac{1}{2}\ln^2\frac{Q^2}{m^2} + \frac{3}{2}\ln\frac{Q^2}{m^2} + \frac{\pi^2}{6} - 2 \right] \ . \end{aligned} \tag{3.8}$$

4 One photon emission

For one photon emission, when the states $|X\rangle$ in (2.5) are states of an electron with momentum l' and photon with momentum k, we have

$$W_{\mu\nu}(l,q) = -\frac{e^2}{8\pi}\int K_{\mu\nu}(2\pi)^4\delta^{(4)}(q+l-l'-k)\frac{d^3l'}{(2\pi)^3 2E_l'}\frac{d^3k}{(2\pi)^3 2\omega} \ , \tag{4.1}$$

where

$$K_{\mu\nu} = g^{\rho\sigma} tr[(\hat{l}'+m)L_{\mu\rho}(\hat{l}+m)\gamma^0 L^\dagger_{\nu\sigma}\gamma^0] \ ,$$

$$L_{\mu\rho} = \gamma_\mu\frac{\hat{l}-\hat{k}+m}{-2\kappa}\gamma_\rho + \gamma_\rho\frac{\hat{l}'+\hat{k}+m}{2\kappa'}\gamma_\mu \ , \ \kappa = (kl) \ , \ \ \kappa' = (kl') = Q^2\frac{1-x}{2x} \ . \tag{4.2}$$

Moving on to integration over κ, we obtain from (4.1) for the convolutions (2.8) of $W_{\mu\nu}(l,q)$

$$W_i(lq,q^2) = -\frac{\alpha}{8\pi}\int_{\kappa_-}^{\kappa_+}\frac{d\kappa}{\sqrt{I_C}}A_i\ , \tag{4.3}$$

where

$$I_C = (Q^2+2\kappa')^2+4m^2Q^2\ ,\quad A_g = K_{\mu\nu}g^{\mu\nu}\ ,\quad A_l = K_{\mu\nu}l^\mu l^\nu\ , \tag{4.4}$$

and

$$\kappa_\pm = \frac{\kappa'}{2(m^2+2\kappa')}\left(2m^2+2\kappa'+Q^2\pm\sqrt{I_C}\right)\ . \tag{4.5}$$

The integration limit κ_- (κ_+) corresponds to forward (backward) virtual Compton scattering.

Direct calculation of the convolutions A_i (4.4) with use of (4.2) gives

$$A_g = 4\left(\frac{m^2}{\kappa'^{\,2}}+\frac{m^2}{\kappa^2}-\frac{2m^2+Q^2}{\kappa\kappa'}+\frac{2}{\kappa'}-\frac{2}{\kappa}\right)(2m^2-Q^2)+8\left(\frac{\kappa}{\kappa'}+\frac{\kappa'}{\kappa}\right)\ , \tag{4.6}$$

$$\begin{aligned} A_l = 2m^2\Bigg[&\left(\frac{m^2}{\kappa'^{\,2}}+\frac{m^2}{\kappa^2}-\frac{2m^2+Q^2}{\kappa\kappa'}+\frac{2}{\kappa'}-\frac{6}{\kappa}\right)(4m^2+Q^2) \\ &+\frac{4m^2\kappa'}{\kappa^2}+\frac{2(2m^2+\kappa)}{\kappa'}+\frac{2(4m^2-3\kappa')}{\kappa}+4\Bigg]-4(Q^2+2\kappa'-2\kappa)\ . \end{aligned} \tag{4.7}$$

Note that A_i should be obtained from Eqs.(7.39) of [32] with the substitutions

$$w^2\to m^2+2\kappa'\ ,\quad q^2\to m^2-2\kappa\ ,\quad \Delta^2\to -Q^2\ ,\quad \Delta_1^2\to 0\ . \tag{4.8}$$

Unfortunately, in the expression for $A_2^{(\frac{1}{2})}$ in Eqs.(7.39) there is a misprint; it contains the extra term $8(p_1p_2)m^4/(q^2-m^2)^2$.

Calculation of $W_i(lq,q^2)$ (4.3) is performed using the integrals

$$\begin{aligned} &\int_{\kappa_-}^{\kappa_+}\frac{d\kappa}{\sqrt{I_C}} = \frac{\kappa'}{m^2+2\kappa'}\ , \qquad \int_{\kappa_-}^{\kappa_+}\frac{\kappa d\kappa}{\sqrt{I_C}} = \frac{\kappa'^2(2m^2+2\kappa'+Q^2)}{2(m^2+2\kappa')^2}\ , \\ &\int_{\kappa_-}^{\kappa_+}\frac{d\kappa}{\kappa^2\sqrt{I_C}} = \frac{1}{m^2\kappa'}\ , \qquad \int_{\kappa_-}^{\kappa_+}\frac{d\kappa}{\kappa\sqrt{I_C}} = \mathcal{L}\ , \end{aligned} \tag{4.9}$$

where

$$\mathcal{L} = \frac{1}{\sqrt{(Q^2+2\kappa')^2+4m^2Q^2}}\ln\left(\frac{2m^2+2\kappa'+Q^2+\sqrt{I_C}}{2m^2+2\kappa'+Q^2-\sqrt{I_C}}\right)\ . \tag{4.10}$$

It gives

$$\begin{aligned} W_g = -\frac{\alpha}{2\pi}\Bigg[&(Q^2-2m^2)\Big[\Big(\frac{Q^2+2m^2}{\kappa'}+2\Big)\mathcal{L}-\Big(\frac{m^2}{\kappa'^2}+\frac{2}{\kappa'}\Big)\frac{\kappa'}{m^2+2\kappa'}-\frac{1}{\kappa'}\Big] \\ &+\frac{\kappa'(2m^2+2\kappa'+Q^2)}{(m^2+2\kappa')^2}+2\kappa'\mathcal{L}\Bigg]\ , \end{aligned} \tag{4.11}$$

$$W_l = -\frac{\alpha}{2\pi}\left[\frac{m^2}{2}\left(8m^2 - 6\kappa' - (Q^2+4m^2)\Big(\frac{Q^2+2m^2}{\kappa'} + 6\Big)\right)\mathcal{L} + m^2\Big(2 + \frac{Q^2+4m^2}{2\kappa'}\Big)\right.$$
$$\left. + \left(m^2\Big((Q^2+4m^2)(\frac{m^2}{\kappa'^2} + \frac{2}{\kappa'}) + \frac{4m^2}{\kappa'} + 6\Big) - Q^2 - 2\kappa'\right)\frac{\kappa'}{2(m^2+2\kappa')}\right] . \tag{4.12}$$

The region of variation of x at fixed Q^2 is determined by the conditions $M_X^2 = (m^2 + 2\kappa') \geq m^2$ and $(lq) \leq E_l q_0 + \sqrt{E_l^2 - m^2}\sqrt{q_0^2 + Q^2}$ with $q_0 = M - E_p' = -Q^2/(2M)$, i.e

$$1 \geq x \geq x_- \ , \quad x_- = \frac{MQ^2}{\sqrt{E_l^2 - m^2}\sqrt{Q^2(4M^2+Q^2)} - E_l Q^2} \ . \tag{4.13}$$

But expressions (4.11) and (4.12) can not be used arbitrarily close to $x = 1$ (i.e. for sufficiently small κ') because of the infrared divergency. The divergency must be regularised in the same way as in the vertex correction (3.8), i.e. by the photon mass λ. Taking into account the photon mass changes both the measure and the limits of integration in (4.3):

$$I_C \to I_C(\lambda) = (Q^2 + 2\kappa' + \lambda^2)^2 + 4m^2Q^2 \ , \tag{4.14}$$

$$\kappa_\pm \to= \kappa_\pm(\lambda) = \frac{(\kappa' + \lambda^2)(2m^2 + 2\kappa' + Q^2 + \lambda^2) \pm \sqrt{(\kappa'^{\,2} - m^2\lambda^2)I_C(\lambda)}}{2(m^2 + 2\kappa' + \lambda^2)} \ . \tag{4.15}$$

In the region $m^2 \gg \kappa' > m\lambda$ at $\lambda \to 0$ they can be taken as

$$I_0 = Q^2(Q^2 + 4m^2) \ , \tag{4.16}$$

$$\kappa_\pm^0 = \frac{\left(\kappa'(2m^2 + Q^2) \pm \sqrt{(\kappa'^{\,2} - m^2\lambda^2)I_0}\right)}{2m^2} \ . \tag{4.17}$$

The singular terms in A_i are

$$A_g = 4(2m^2 - Q^2)\left(\frac{m^2}{\kappa'^{\,2}} + \frac{m^2}{\kappa^2} - \frac{2m^2 + Q^2}{\kappa\kappa'}\right) \ , \tag{4.18}$$

$$A_l = 2m^2(4m^2 + Q^2)\left(\frac{m^2}{\kappa'^{\,2}} + \frac{m^2}{\kappa^2} - \frac{2m^2 + Q^2}{\kappa\kappa'}\right) \ . \tag{4.19}$$

Corresponding integrals become

$$\int_{\kappa_-^0}^{\kappa_+^0} \frac{d\kappa}{\sqrt{I_0}} = \frac{\sqrt{(\kappa'^{\,2} - m^2\lambda^2)}}{m^2} \ , \qquad \int_{\kappa_-^0}^{\kappa_+^0} \frac{d\kappa}{\kappa^2\sqrt{I_0}} = \frac{4\kappa'\sqrt{(\kappa'^{\,2} - m^2\lambda^2)}}{4m^2\kappa'^2 + \lambda^2Q^2(4m^2 + Q^2)} \ ,$$
$$\int_{\kappa_-^0}^{\kappa_+^0} \frac{d\kappa}{\kappa\sqrt{I_0}} = \mathcal{L}_0 \ , \tag{4.20}$$

where

$$\mathcal{L}_0 = \frac{1}{\sqrt{Q^2(4m^2 + Q^2)}} \ln\left(\frac{\kappa'(2m^2 + Q^2) + \sqrt{(\kappa'^2 - m^2\lambda^2)Q^2(4m^2 + Q^2)}}{\kappa'(2m^2 + Q^2) - \sqrt{(\kappa'^2 - m^2\lambda^2)Q^2(4m^2 + Q^2)}}\right) \ . \tag{4.21}$$

It gives for W_i (4.3) in the region $m^2 \gg \kappa' > m\lambda$

$$W_g = \frac{2m^2 - Q^2}{2}\frac{dw}{d\kappa'} \ , \quad W_l = m^2\frac{4m^2 + Q^2}{4}\frac{dw}{d\kappa'} \ , \tag{4.22}$$

where $mdw/d\kappa'$ is the spectral probability density for soft photon emission with account of photon mass in the rest frame of the final electron,

$$\frac{dw}{d\kappa'} = \frac{\alpha}{\pi}\frac{1}{\kappa'}\left[(2m^2 + Q^2)\mathcal{L}_0 - -\frac{\sqrt{\kappa'^2 - m^2\lambda^2}}{\kappa'} - \frac{4m^2\kappa'\sqrt{\kappa'^2 - m^2\lambda^2}}{4m^2\kappa'^2 + \lambda^2 Q^2(4m^2 + Q^2)}\right] . \tag{4.23}$$

Using (2.9), (2.10) and (4.22), one obtains

$$F_2(x, Q^2) = 2F_1(x, Q^2) = \frac{Q^2}{2}\frac{dw}{d\kappa'} \ , \tag{4.24}$$

so that (2.14) gives for the soft photon emission cross section

$$\frac{d\sigma^{\gamma}_{soft}}{dQ^2 dx} = \frac{d\sigma^B}{dQ^2}\frac{Q^2}{2}\frac{dw}{d\kappa'} \ . \tag{4.25}$$

Integration (4.25) over the region $\kappa_0 > \kappa' > m\lambda$ $(1 - m\lambda/Q^2 > x > 1 - 2\kappa_0/Q^2 \ , \quad dx = -2d\kappa'/Q^2)$ at $\kappa_0 \ll m^2, \ \kappa_0 \ll Q^2$ provides at $\lambda \to 0$

$$\frac{d\sigma^{\gamma}_{soft}}{dQ^2} = \frac{d\sigma_B}{dQ^2}\delta^e_{soft} \ , \tag{4.26}$$

$$\begin{aligned}\delta^e_{soft} = \frac{\alpha}{\pi}\Bigg[\frac{1}{\beta}\Bigg(&\ln\left(\frac{(1+\beta)}{(1-\beta)}\right)\left(\ln\left(\frac{2\kappa_0}{m\lambda}\right) + \frac{1}{2}\right) + \frac{1}{2}\mathrm{Li}_2\left(1 - \frac{(1+\beta)}{(1-\beta)}\right)\\ &- \frac{1}{2}\mathrm{Li}_2\left(1 - \frac{(1-\beta)}{(1+\beta)}\right)\Bigg) - 2\ln\left(\frac{2\kappa_0}{m\lambda}\right) + 1\Bigg] \ ,\end{aligned} \tag{4.27}$$

where β is given by (3.7).

At $Q^2 \gg m^2$ one has

$$\delta^e_{soft} = \frac{\alpha}{\pi}\left[2\ln\left(\frac{2\kappa_0}{m\lambda}\right)\left(\ln\left(\frac{Q^2}{m^2}\right) - 1\right) - \ln^2\left(\frac{Q^2}{m^2}\right) + \ln\left(\frac{Q^2}{m^2}\right) - \frac{\pi^2}{6} + 1\right] , \tag{4.28}$$

which together with (3.8) gives

$$\delta^e_{vertex} + \delta^e_{soft} = \frac{\alpha}{\pi}\left[2\ln\left(\frac{2\kappa_0}{m^2}\right)\left(\ln\left(\frac{Q^2}{m^2}\right) - 1\right) - \frac{3}{2}\ln^2\left(\frac{Q^2}{m^2}\right) + \frac{5}{2}\ln\left(\frac{Q^2}{m^2}\right) - 1\right] . \tag{4.29}$$

As it should be, the dependence on λ disappeared in the sum of corrections from elastic scattering and soft photon emission.

To find the contribution of real photons with $\kappa' > \kappa_0$ for arbitrary Q^2 is not so easy. In the following we restrict ourselves to considering the case $Q^2 \gg m^2$. In this case it is possible to introduce the intermediate scale κ_1, such that $Q^2 \gg \kappa_1 \gg m^2$, and calculate the contributions of the regions $\kappa' < \kappa_1$ and $\kappa' > \kappa_1$, simplifying the integrands in them as it is described in Appendix B. In the sum of these contributions dependence on the intermediate scale disappears (see (B.17)). The remaining dependence on the boundary κ_0 between soft and hard emission vanishes in the sum of the correction (B.17) due to the hard emission with $\delta^e_{vertex} + \delta^e_{soft}$ (4.29), so that for the total correction δ^e one has in the one-loop approximation at $Q^2 \gg m^2$

$$\begin{aligned}
\delta^e_{one-loop} = \frac{\alpha}{2\pi}\Bigg[&\big(x_- + \frac{x_-^2}{2} + 2\ln(1-x_-)\big)\ln\frac{Q^2}{m^2} - 2\ln x_- \ln(1-x_-) \\
&- \ln^2(1-x_-) - \frac{x_-}{2}(2+x_-)\ln x_- - (2 + x_- + \frac{x_-^2}{2})\ln(1-x_-) \\
&- 2\mathrm{Li}_2(x_-) - \frac{3}{2}x_- - x_-^2 \\
&+ \frac{Q^2}{4(pl)}\frac{Q^2G_M^2 + 4M^2G_E^2}{(4M^2+Q^2)R(1,Q^2)}\Big[(2\ln x_- - (1-x_-^2))\ln\frac{Q^2}{m^2} - \ln^2 x_- \\
&+ (1-x_-^2)\ln(x_-(1-x_-)) + (1-x_-)(3+2x_-) - \frac{\pi^2}{3} + 2\mathrm{Li}_2(x_-)\Big] \\
&+ \frac{Q^2}{16(pl)^2}\frac{Q^2(Q^2+2M^2)G_M^2 - 8M^4G_E^2}{(4M^2+Q^2)R(1,Q^2)}\Bigg[\left(\frac{(1-x_-^2)(2+x_-)}{x_-} - 2\ln x_-\right)\ln\frac{Q^2}{m^2} \\
&+ \ln^2 x_- - \frac{(1-x_-^2)(2+x_-)}{x_-}\ln(1-x_-) \\
&- \left(\frac{2}{x_-} + 1 - 2x_- - x_-^2\right)\ln x_- - (1-x_-)(\frac{1}{x_-} + 5 + 2x_-) + \frac{\pi^2}{3} - 2\mathrm{Li}_2(x_-)\Bigg] \\
&+ \frac{Q^2}{4(pl)^2}\frac{Q^2G_M^2 - 2M^2G_E^2}{R(1,Q^2)}\ln x_-\Bigg] .
\end{aligned} \tag{4.30}$$

The only approximation used here is $Q^2 \gg m^2$.

The correction is strongly simplified at small x_- (i.e. at small Q^2), when we have

$$\begin{aligned}
\delta^e_{one-loop} = \frac{\alpha}{2\pi(1+2\rho x_-)}\Bigg[&\Big(-2x_- - x_-^2\big(1 - \ln x_- + \rho(3 - 2\ln x_-)\big)\Big)\ln\left(\frac{Q^2}{m^2}\right) \\
&+ x_-(2\ln x_- - 1) + x_-^2\left(-\frac{1}{2}\ln^2 x_- - \ln x_- - \frac{\pi^2}{6} + \frac{1}{2} + \right. \\
&\left.\rho\big(-\ln^2 x_- + 3\ln x_- - \frac{\pi^2}{3}\big)\right)\Bigg] ,
\end{aligned} \tag{4.31}$$

where $\rho = (pl)/M^2$. As we can see, only the terms with power smallness in x_- remain in the full correction. The terms that do not have such smallness cancel out not only if they are strengthened by powers of $\ln(Q^2/m^2)$, but also without such strengthening, as it was noted in [22].

5 Parton picture

In the parton picture the structure functions $F_i(x,Q^2)$ are expressed through parton distributions. In the leading logarithmic approximation (LLA)

$$F_2(x,Q^2) = 2xF_1(x,Q^2) = x(f_e^e(x,Q^2) + f_e^{\bar{e}}(x,Q^2)) \ , \tag{5.1}$$

where $f_e^e(x,Q^2)$ and $f_e^{\bar{e}}(x,Q^2)$ are the electron and positron distributions in the initial electron and the first equality is the Callan-Gross relation [33], arising from the fact that the partons have spin 1/2.

In the LLA the parton distributions can be calculated using the equations [27, 28]

$$\frac{df_e^a(x,Q^2)}{d\ln Q^2} = \frac{\alpha(Q^2)}{2\pi}\sum_b \int_x^1 \frac{dz}{z} P_b^a(\frac{x}{z}) f_e^b(z,Q^2), \tag{5.2}$$

where $a, b = e, \bar{e}, \gamma$, $P_b^a(z)$ are the splitting functions,

$$\begin{aligned} P_e^\gamma(z) = P_{\bar{e}}^\gamma(z) &= \frac{1+(1-z)^2}{z}, \qquad P_\gamma^e(z) = P_\gamma^{\bar{e}}(z) = z^2 + (1-z)^2 \ , \\ P_e^e(z) = P_{\bar{e}}^{\bar{e}}(z) &= \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \ . \end{aligned} \tag{5.3}$$

Here the generalized function $\frac{1}{(1-z)_+}$ is defined by the relation

$$\int_0^1 \frac{f(z)}{(1-z)_+} dz = \int_0^1 \frac{f(z)-f(1)}{(1-z)} dz \ . \tag{5.4}$$

The evolution equations must be complemented by the initial conditions, which can be taken as

$$f_e^a(x,m^2) = \delta_e^a \delta(1-x) \ . \tag{5.5}$$

Presenting parton distributions as the sum of the distributions of valence (v) and sea (s) partons in electron

$$f_e^e(x,Q^2) = f_e^v(x,Q^2) + f_e^s(x,Q^2) \ , \quad f_e^{\bar{e}}(x,Q^2) = f_e^s(x,Q^2) \ , \tag{5.6}$$

we obtain that these distributions obey the equations

$$\frac{df_e^v(x,Q^2)}{d\ln Q^2} = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} P_e^e(\frac{x}{z}) f_e^v(z,Q^2), \tag{5.7}$$

$$\frac{df_e^s(x,Q^2)}{d\ln Q^2} = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \left(P_e^e(\frac{x}{z}) f_e^s(z,Q^2) + P_\gamma^e(\frac{x}{z}) f_e^\gamma(z,Q^2) \right), \tag{5.8}$$

with initial

$$f_e^v(x,m^2) = \delta(1-x) \ , \ f_e^s(x,m^2) = 0 \tag{5.9}$$

and charge conservation

$$\int_0^1 f_e^v(x,Q^2)dx = 1 \tag{5.10}$$

conditions. Writing with the two-loop accuracy

$$f_e^v(x,Q^2) = \delta(1-x) + \frac{\alpha}{2\pi} L\, V_1(x) + \left(\frac{\alpha}{2\pi}\right)^2 \frac{L^2}{2}\, V_2(x)\ , \tag{5.11}$$

$$f_e^s(x,Q^2) = \left(\frac{\alpha}{2\pi}\right)^2 \frac{L^2}{2} S_2(x)\ , \tag{5.12}$$

where $L = \ln(Q^2/m^2)$, and taking into account in $\alpha(Q^2)$ only the one-loop correction coming from vacuum polarisation by electrons

$$\alpha(Q^2) = \alpha + \frac{\alpha^2}{3\pi} L\ , \tag{5.13}$$

we have

$$V_1(x) = P_e^e(x) = \frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\ , \tag{5.14}$$

$$V_2(x) = \frac{2}{3}\, P_e^e(x) + \int_x^1 \frac{dz}{z} P_e^e\left(\frac{x}{z}\right) P_e^e\,(z) = \frac{2}{3} P_e^e(x) + 8\left(\frac{\ln(1-x)}{(1-x)}\right)_+$$

$$+ \left.\frac{1+4x+x^2}{(1-x)}\right._+ - \frac{1+3x^2}{(1-x)}\ln x - 4(1+x)\ln(1-x) - \left(\frac{2\pi^2}{3} - \frac{9}{4}\right)\delta(1-x)\ , \tag{5.15}$$

$$S_2(x) = \int_x^1 \frac{dz}{z} P_e^\gamma\left(\frac{x}{z}\right) P_\gamma^e\,(z) = 2(1+x)\ln x + 1 - x + \frac{4(1-x^3)}{3x}\ , \tag{5.16}$$

where the generalized function $\left(\frac{\ln(1-z)}{(1-z)}\right)_+$ are defined by the relation analogous to (5.4)

$$\int_0^1 \left(\frac{\ln(1-z)}{(1-z)}\right)_+ f(z)dz = \int_0^1 \left(\frac{\ln(1-z)}{(1-z)}\right)(f(z) - f(1))dz\ . \tag{5.17}$$

The coefficients of the delta-function terms in (5.14), (5.15) are determined by the requirements

$$\int_0^1 dx V_i(x) = 0 \tag{5.18}$$

following from the charge conservation condition (5.10).

Writing the cross section (2.14) at $Q^2 \gg m^2$, $F_2(x,Q^2) = 2xF_1(x,Q^2)$ as

$$\frac{d\sigma}{dQ^2dx} = \frac{4\pi\left(\alpha(Q^2)\right)^2}{Q^4}\frac{F_2(x,Q^2)}{x} R(x,Q^2)\ , \tag{5.19}$$

where $R(x,Q^2)$ is given by (2.21), we have for the radiative correction δ^e in the leading logarithmic approximation

$$\delta^e_{LLA} = \int_{x_-}^1 dx \frac{R(x,Q^2)}{R(1,Q^2)}(f_e^v(x,Q^2) + 2f_e^s(x,Q^2)) - 1\ , \tag{5.20}$$

This representation permits to find the radiative correction δ^e in any order of perturbation theory.

With the two-loop accuracy $f_e^v(x,Q^2)$ and $f_e^s(x,Q^2)$ are given by Eqs. (5.11) and (5.12) respectively. Using (2.21) and (5.10), we have

$$\delta^e_{LLA} = -\int_0^{x_-} dx f_e^v(x,Q^2) + \int_{x_-}^1 dx \left[\left(\frac{R(x,Q^2)}{R(1,Q^2)} - 1 \right) f_e^v(x,Q^2) + 2\frac{R(x,Q^2)}{R(1,Q^2)} f_e^s(x,Q^2) \right] . \tag{5.21}$$

In the one-loop approximation only $f_e^v(x,Q^2)$ does contribute. Simple integration gives

$$\begin{aligned} \delta^e_{one-loop,LLA} = \frac{\alpha}{2\pi} L \Bigg[& 2\ln(1-x_-) + x_- + \frac{x_-^2}{2} + \frac{Q^2}{2(pl)} \frac{Q^2 G_M^2 + 4M^2 G_E^2}{(4M^2+Q^2)R(1,Q^2)} \Bigg(\ln x_- \\ & - \frac{1-x_-^2}{2} \Bigg) + \frac{Q^2}{8(pl)^2} \frac{Q^2(2M^2+Q^2)G_M^2 - 8M^4 G_E^2}{(4M^2+Q^2)R(1,Q^2)} \Bigg(-\ln x_- \\ & + (1-x_-)\left(\frac{1}{x_-} + \frac{3}{2} + \frac{x_-}{2} \right) \Bigg) \Bigg] , \end{aligned} \tag{5.22}$$

in accordance with (4.30).

The two-loop correction contains contributions of both f_e^v and f_e^s. Using (5.11)-(5.16) and (2.21), one obtains from (5.21) the two-loop contribution in the form

$$\begin{aligned} \delta^e_{two-loop,LLA} = & \left(\frac{\alpha}{2\pi} \right)^2 \frac{L^2}{2} \Bigg[-\int_0^{x_-} dx V_2(x) + \frac{Q^2 G_M^2 + 4M^2 G_E^2}{(4M^2+Q^2)R(1,Q^2)} \\ & \times \int_{x_-}^1 dx \Bigg(2S_2(x) - \frac{Q^2}{2(pl)} \left(\frac{1-x}{x} V_2(x) + \frac{2}{x} S_2(x) \right) \Bigg) \\ & + \frac{Q^2}{8(pl)^2} \frac{Q^2(2M^2+Q^2)G_M^2 - 8M^4 G_E^2}{(4M^2+Q^2)R(1,Q^2)} \int_{x_-}^1 dx \left(\frac{1-x^2}{x^2} V_2(x) + \frac{2}{x^2} S_2(x) \right) \Bigg) \Bigg] . \end{aligned} \tag{5.23}$$

Elementary integration gives

$$\begin{aligned} \int_0^{x_-} dx\, V_2(x) = & \, 4\mathrm{Li}_2(x_-) - 4\ln(1-x_-)\ln\frac{(1-x_-)}{x_-} \\ & - (\frac{4}{3} + 4x_- + 2x_-^2)\ln(1-x_-) + 3x_-(1+\frac{x_-}{2})\ln x_- - \frac{8}{3}x_- - \frac{7}{12}x_-^2 , \end{aligned} \tag{5.24}$$

$$\begin{aligned} \int_{x_-}^1 dx \left(\frac{1}{x} - 1 \right) V_2(x) = & \, 4\mathrm{Li}_2(x_-) - \frac{2}{3}\pi^2 + \frac{1}{2}\ln^2 x_- + 2(1-x_-^2)\ln(1-x_-) \\ & - (\frac{5}{3} - \frac{3}{2}x_-^2)\ln x_- + \frac{1}{12}(1-x_-)(31+7x_-) , \end{aligned} \tag{5.25}$$

$$\begin{aligned} \int_{x_-}^1 dx \left(\frac{1}{x^2} - 1 \right) V_2(x) = & \, 4\mathrm{Li}_2(x_-) - \frac{2}{3}\pi^2 + \frac{1}{2}\ln^2 x_- \\ & + 2(1-x_-^2)(\frac{2}{x_-} + 1)\ln(1-x_-) - (\frac{1}{x_-} + \frac{5}{3} - 3x_- - \frac{3}{2}x_-^2)\ln x_- \\ & + (1-x_-)(\frac{2}{3x_-} + \frac{13}{4} + \frac{7}{12}x_-) , \end{aligned} \tag{5.26}$$

$$\int_{x_-}^{1} dx S_2(x) = -(\frac{4}{3} + 2x_- + x_-^2)\ln x_- - \frac{1}{9}(1-x_-)(22+13x_- + 4x_-^2)\ , \tag{5.27}$$

$$\int_{x_-}^{1} dx \frac{S_2(x)}{x} = -\ln^2 x_- - (2x_- + 1)\ln x_- + \frac{(1-x_-)}{3}(\frac{4}{x_-} - 11 - 2x_-)\ , \tag{5.28}$$

$$\int_{x_-}^{1} dx \frac{S_2(x)}{x^2} = -\ln^2 x_- + (\frac{2}{x_-} + 1)\ln x_- + \frac{(1-x_-)}{3}(\frac{2}{x_-^2} + \frac{11}{x_-} - 4)\ , \tag{5.29}$$

so that

$$\begin{aligned} \delta^e_{two-loop,LLA} = &\left(\frac{\alpha}{2\pi}\right)^2 \frac{L^2}{2}\Bigg[-4\mathrm{Li}_2(x_-) + 4\ln(1-x_-)\ln\frac{(1-x_-)}{x_-} \\ &+ \left(\frac{4}{3} + 4x_- + 2x_-^2\right)\ln(1-x_-) - 3x_-(1+\frac{x_-}{2})\ln x_- + \frac{8}{3}x_- + \frac{7}{12}x_-^2 \\ &+ \frac{Q^2G_M^2 + 4M^2G_E^2}{(4M^2+Q^2)R(1,Q^2)}\Bigg(-2\left(\frac{4}{3} + 2x_- + x_-^2\right)\ln x_- \\ &- \frac{2}{9}(1-x_-)(22+13x_-+4x_-^2) - \frac{Q^2}{2(pl)}\Bigg(4\mathrm{Li}_2(x_-) - \frac{2}{3}\pi^2 - \frac{3}{2}\ln^2 x_- \\ &+ 2(1-x_-^2)\ln(1-x_-) - (\frac{11}{3} + 4x_- - \frac{3}{2}x_-^2)\ln x_- \\ &+ (1-x_-)(\frac{8}{3x_-} - \frac{19}{4} - \frac{3}{4}x_-)\Bigg)\Bigg) + \frac{Q^2}{8(pl)^2}\frac{Q^2(2M^2+Q^2)G_M^2 - 8M^4G_E^2}{(4M^2+Q^2)R(1,Q^2)} \\ &\times\Bigg(4\mathrm{Li}_2(x_-) - \frac{2}{3}\pi^2 - \frac{3}{2}\ln^2 x_- + 2(1-x_-^2)(\frac{2}{x_-}+1)\ln(1-x_-) \\ &+ (\frac{3}{x_-} + \frac{1}{3} + 3x_- + \frac{3}{2}x_-^2)\ln x_- + (1-x_-)(\frac{4}{3x_-^2} + \frac{8}{x_-} + \frac{7}{12}(1+x_-))\Bigg)\Bigg]\ . \end{aligned} \tag{5.30}$$

Note that at small momentum transfer, i.e. at small x_-, the valence quark contribution is suppressed as well as in the one loop due to the charge conservation requirement (5.18). It is not so for the sea quark contribution [22]. The sea quark distribution is singular at $x=0$ and the lower limit x_0 of the integration in (5.27) can not be taken equal to 0. Therefore the two-loop correction is not suppressed at small momentum transfer for experimental conditions at which production of electron-positron pairs is not forbidden. For such conditions we have at $x_- \ll 1$

$$\delta^e_{two-loop,LLA} = \left(\frac{\alpha}{2\pi}\right)^2 \frac{L^2}{2}\left[-\frac{8}{3}\ln x_- - \frac{44}{9} - \frac{4}{3(1+2\rho x_-)}\right]\ . \tag{5.31}$$

At first glance it seems that more preferable are the conditions at which production of electron-positron pairs is forbidden. In this case f_e^s must be omitted in Eq. (5.21), and the term with $P_e^e(x)$ must be omitted in its expression for $V_2(x)$ in (5.15). However, this is not the whole truth. The term with $P_e^e(x)$ in $V_2(x)$ meets contributions from not only real, but also virtual pairs, which can not be suppressed, and therefore their contribution must be restored. It means that the term

$$\left(\frac{\alpha}{\pi}\right)^2\left[-\frac{1}{36}L^3+\frac{19}{72}L^2\right]\delta(1-x) \tag{5.32}$$

must be added to f_e^v (see for details [30]). Therefore, in this case

$$\begin{aligned}
\delta^e_{two-loop,LLA} &= \left(\frac{\alpha}{2\pi}\right)^2\frac{L^2}{2}\Bigg[-\frac{2}{9}L+\frac{19}{9}-4\mathrm{Li}_2(x_-)+4\ln(1-x_-)\ln\frac{(1-x_-)}{x_-} \\
&+2x_-(2+x_-)\ln(1-x_-)-3x_-(1+\frac{x_-}{2})\ln x_-+2x_-+\frac{1}{4}x_-^2 \\
&-\frac{Q^2}{2(pl)}\frac{Q^2G_M^2+4M^2G_E^2}{(4M^2+Q^2)R(1,Q^2)}\bigg(4\mathrm{Li}_2(x_-)-\frac{2}{3}\pi^2+\frac{1}{2}\ln^2 x_- \\
&+2(1-x_-^2)\ln(1-x_-)-(1-\frac{3}{2}x_-^2)\ln x_-+\frac{1}{4}(1-x_-)(9+x_-)\bigg) \\
&+\frac{Q^2}{8(pl)^2}\frac{Q^2(2M^2+Q^2)G_M^2-8M^4G_E^2}{(4M^2+Q^2)R(1,Q^2)}\bigg(4\mathrm{Li}_2(x_-)-\frac{2}{3}\pi^2+\frac{1}{2}\ln^2 x_- \\
&+2(1-x_-^2)(\frac{2}{x_-}+1)\ln(1-x_-)-(\frac{1}{x_-}+1-3x_-+\frac{3}{2}x_-^2)\ln x_- \\
&+\frac{1}{4}(1-x_-)(9+x_-)\bigg)\Bigg].
\end{aligned} \tag{5.33}$$

The relative magnitude of the corrections (5.23) and (5.33) depends on energy and momentum transfer.

6 Conclusion

As it was shown in [22], the setting of the experiment with recoil proton detection suggested by A.A. Vorobev [12] for measurement of proton radius, has an interesting feature – cancellation of main radiative corrections. Here we calculated radiative corrections to the cross section of electron-proton scattering for experiments of this kind in a wide range of kinematic parameters using the method of structure functions and parton distributions. We calculated the one-loop corrections due to electron interaction for momentum transfer Q limited only by the requirement $Q \gg m$, m being the electron mass, and proved that when at small Q the cancellation of the virtual and real radiative corrections has a power accuracy.

In the two-loop approximation we calculated these corrections with logarithmic accuracy, again for momentum transfer Q limited only by the requirement $Q \gg m$, using the parton distribution method [25]-[28] developed for the theoretical description of the deep inelastic electron-proton scattering and adopted in [30] for calculation of radiative corrections in QED. We calculated the radiation corrections both for such an experiment setup when the production of additional electron-positron pairs is allowed, and for such when it is forbidden.

Appendix A

The vacuum polarisation $\mathcal{P}(q^2)$ contains lepton (electron, muon, τ-lepton) and hadron contributions:

$$\mathcal{P}(q^2) = \mathcal{P}_e(q^2) + \mathcal{P}_\mu(q^2) + \mathcal{P}_\tau(q^2) + \mathcal{P}_h(q^2). \tag{A.1}$$

One-loop lepton contribution $\mathcal{P}_l^{(1)}(q^2)$, $(l = e, \mu, \tau)$ is well known (see, for example, [31]):

$$\mathcal{P}_l^{(1)}(q^2) = \frac{\alpha}{\pi}\left(\frac{1}{3}\sqrt{1-\frac{4\,m_l^2}{q^2}}\left(1+\frac{2\,m_l^2}{q^2}\right)\ln\left(\frac{\sqrt{1-\frac{4m_l^2}{q^2}}+1}{\sqrt{1-\frac{4m_l^2}{q^2}}-1}\right)-\frac{4\,m_l^2}{3\,q^2}-\frac{5}{9}\right). \tag{A.2}$$

At $Q^2 = -q^2 \gg 4m_l^2$

$$\mathcal{P}_l^{(1)}(q^2) = \frac{\alpha}{3\pi}\left(\ln\left(\frac{Q^2}{m_l^2}\right)-\frac{5}{3}\right) , \tag{A.3}$$

and at $Q^2 = -q^2 \ll 4m_l^2$

$$\mathcal{P}_l(q^2) = \frac{\alpha}{15\pi}\frac{Q^2}{m_l^2} . \tag{A.4}$$

The lepton contributions are known also in higher orders of perturbation theory (see, for example, [34, 35]). For us it is enough to know that the two-loop contribution contains only the first degree of $\ln\left(\frac{Q^2}{m_l^2}\right)$.

The hadron contribution $\mathcal{P}_h(q^2)$ is expressed in terms of the total cross section of one-photon electron-positron pair annihilation into hadrons

$$\mathcal{P}_h(q^2) = \frac{q^2}{4\pi^2\alpha}\int_{4m_\pi^2}^{\infty} ds\frac{\sigma_{e^+e^-\to hadrons}(s)}{s-q^2} . \tag{A.5}$$

This contribution is small compared with α/π at $Q^2 < 4m_\pi^2$, becomes of order of α/π only at $Q^2 \sim 4m_\pi^2$ and then grows logarithmically with Q^2. Recent review is given in [35].

Appendix B

To find the contribution δ^e_{hard} of the one-photon emission with $\kappa' > \kappa_0$ to the radiative correction δ^e at $Q^2 \gg m^2$ it is convenient to introduce the intermediate scale κ_1 such that $Q^2 \gg \kappa_1 \gg m^2$. In the region $\kappa_1 > \kappa' > \kappa_0$ one can put

$$W_1 = -\frac{\alpha}{2\pi}\left[\frac{Q^2}{\kappa'}\left(\ln\left(\frac{Q^4}{m^2(m^2+2\kappa')}\right)-2\right)+\frac{Q^2\kappa'}{(m^2+2\kappa')^2}\right] , \tag{B.6}$$

$$W_2 = 0\ ,\ \ F_1 = -\frac{1}{2}W_1\ ,\ \ F_2 = 2F_1\ . \tag{B.7}$$

Therefore in this region we have from Eqs. (2.20)–(2.22)

$$\frac{d\sigma^\gamma}{dQ^2dx} = -W_1\frac{d\sigma_B}{dQ^2} . \tag{B.8}$$

Using that in this region it is possible to put $\kappa' = Q^2(1-x)/2$, it is easy to obtain the part $\delta^{(1)}_{hard}$ of the correction δ^e_{hard}, defined by Eqs. (2.18) and (2.19), from this region:

$$\begin{aligned}\delta^{(1)}_{hard} &= \frac{\alpha}{\pi}\int_{\kappa_0}^{\kappa_1}\frac{d\kappa'}{\kappa'}\left[2\Big(\ln\left(\frac{Q^2}{m^2}\right)-1\Big)-\ln\left(\frac{m^2+2\kappa'}{m^2}\right)+\frac{\kappa'^2}{(m^2+2\kappa')^2}\right] \\ &= \frac{\alpha}{\pi}\left[2\ln\left(\frac{\kappa_1}{\kappa_0}\right)\left(\ln\left(\frac{Q^2}{m^2}\right)-1\right)-\frac{1}{2}\ln^2\left(\frac{2\kappa_1}{m^2}\right)+\frac{1}{4}\Big(\ln\left(\frac{2\kappa_1}{m^2}\right)-1\Big)-\frac{\pi^2}{6}\right],\end{aligned} \tag{B.9}$$

Using (4.29), we obtain

$$\begin{aligned}\delta^e_{vert}+\delta_{soft}+\delta^{(1)}_{hard} &= \frac{\alpha}{\pi}\left[2\ln\left(\frac{2\kappa_1}{m^2}\right)\left(\ln\left(\frac{Q^2}{m^2}\right)-1\right)-\frac{3}{2}\ln^2\left(\frac{Q^2}{m^2}\right)\right. \\ &\left.+\frac{5}{2}\ln\left(\frac{Q^2}{m^2}\right)-\frac{1}{2}\ln^2\left(\frac{2\kappa_1}{m^2}\right)+\frac{1}{4}\ln\left(\frac{2\kappa_1}{m^2}\right)-\frac{\pi^2}{6}-\frac{5}{4}\right].\end{aligned} \tag{B.10}$$

In the region $\kappa_{max} > \kappa' > \kappa_1$, i.e. $1-\frac{2\kappa_1}{Q^2} > x > x_-$ at $Q^2 \gg m^2$ one can put

$$W_1 = -\frac{\alpha}{2\pi}\Big(\frac{1+x^2}{1-x}\ln\left(\frac{Q^2}{m^2x(1-x)}\right)+\frac{1-8x}{2(1-x)}\Big), \tag{B.11}$$

$$W_2 = \frac{\alpha}{2\pi}\frac{Q^2}{4x}, \tag{B.12}$$

and

$$F_1 = \frac{1}{2}\left(\frac{4x^2\,W_2}{Q^2}-W_1\right), \tag{B.13}$$

$$F_2 = x\left(\frac{12x^2}{Q^2}\,W_2-W_1\right) = 2xF_1+\frac{\alpha}{\pi}x^2. \tag{B.14}$$

As it is seen, the Callan-Gross relation [33] is violated in this region. It could be expected, since this relation is valid only in the collinear approximation.

Using Eqs. (2.20)-(2.22), we have for the part $\delta^{(2)}_{hard}$ of the correction δ^e_{hard} from this region:

$$\begin{aligned}\delta^{(2)}_{hard} = \int_{x_-}^{1-2\frac{\kappa_1}{Q^2}} dx &\left[\frac{F_2(x,Q^2)}{x}-\frac{Q^2}{2(pl)}\frac{Q^2G_M^2+4M^2G_E^2}{(4M^2+Q^2)R(1,Q^2)}\frac{F_2(x,Q^2)(1-x)}{x^2}\right. \\ &+\frac{Q^2}{8(pl)^2}\left(\frac{Q^2(Q^2+2M^2)G_M^2-8M^4G_E^2}{(4M^2+Q^2)R(1,Q^2)}\right)\frac{F_2(x,Q^2)(1-x^2)}{x^3} \\ &\left.-\frac{Q^2}{8(pl)^2}\frac{(Q^2G_M^2-2M^2G_E^2)}{R(1,Q^2)}\frac{(F_2(x,Q^2)-2xF_1(x,Q^2))}{x^3}\right].\end{aligned} \tag{B.15}$$

Note that in the integral (B.15) the upper limit can be set equal to 1 in all terms except the first one. It gives

$$\begin{aligned}
\delta^{(2)}_{hard} = \frac{\alpha}{2\pi}\Bigg[& \left(3\ln\left(\frac{Q^2}{m^2}\right) - 4\ln\left(\frac{2\kappa_1}{m^2}\right) - 5 + x_- + \frac{x_-^2}{2} + 2\ln(1-x_-)\right)\ln\left(\frac{Q^2}{m^2}\right) \\
& + \ln^2\left(\frac{2\kappa_1}{m^2}\right) + \frac{7}{2}\ln\left(\frac{2\kappa_1}{m^2(1-x_-)}\right) - \ln^2(1-x_-) + 2\mathrm{Li}_2(1-x_-) + \frac{5}{2} \\
& - \frac{x_-}{2}(2+x_-)\ln x_- + \frac{(1-x_-)}{2}(3+x_-)\ln(1-x_-) - \frac{x_-}{2}(3+2x_-)\Bigg] \\
& + \frac{Q^2}{4(pl)}\frac{Q^2G_M^2 + 4M^2G_E^2}{4M^2+Q^2}\Bigg((2\ln x_- - (1-x_-^2))\ln\left(\frac{Q^2}{m^2}\right) - \ln^2 x_- - \frac{\pi^2}{3} \\
& + 2\mathrm{Li}_2(x_-) + (1-x_-)(3+2x_-) + (1-x_-^2)\ln(x_-(1-x_-))\Bigg) \\
& + \frac{Q^2}{16(pl)^2}\frac{Q^2(Q^2+2M^2)G_M^2 - 8M^4G_E^2}{4M^2+Q^2} \\
& \times \Bigg(\Big((1-x_-^2)\frac{(2+x_-)}{x_-} - 2\ln x_-\Big)\ln\left(\frac{Q^2}{m^2}\right) + \frac{\pi^2}{3} - 2\mathrm{Li}_2(x_-) + \ln^2 x_- \\
& - (1-x_-^2)\frac{(2+x_-)}{x_-}\ln(1-x_-) - \left(\frac{2}{x_-} + 1 - 2x_- - x_-^2\right)\ln x_- \\
& - (1-x_-)(\frac{1}{x_-} + 5 + 2x_-)\Bigg) + \frac{Q^2}{4(pl)^2}(Q^2G_M^2 - 2M^2G_E^2)\ln x_-\Bigg] .
\end{aligned} \tag{B.16}$$

The intermediate parameter κ_1 disappears in the sum $\delta^e_{hard} = \delta^{(2)}_{hard} + \delta^{(1)}_{hard}$:

$$
\begin{aligned}
\delta^e_{hard} = \frac{\alpha}{2\pi}\Bigg[& \left(3\ln\left(\frac{Q^2}{m^2}\right) - 4\ln\left(\frac{2\kappa_0}{m^2}\right) - 5 + x_- + \frac{x_-^2}{2} + 2\ln(1-x_-)\right)\ln\left(\frac{Q^2}{m^2}\right) \\
& + 4\ln\left(\frac{2\kappa_0}{m^2}\right) - \ln^2(1-x_-) + 2\mathrm{Li}_2(1-x_-) - \frac{x_-}{2}(2+x_-)\ln x_- - \frac{3}{2}x_- \\
& - x_-^2 - (2 + x_- + \frac{x_-^2}{2})\ln(1-x_-) - \frac{\pi^2}{3} + 2 + \frac{Q^2}{4(pl)}\frac{Q^2G_M^2 + 4M^2G_E^2}{(4M^2+Q^2)R(1,Q^2)} \\
& \times\Bigg((2\ln x_- - (1-x_-^2))\ln\left(\frac{Q^2}{m^2}\right) - \ln^2 x_- - \frac{\pi^2}{3} + 2\mathrm{Li}_2(x_-) \\
& + (1-x_-)(3+2x_-) + (1-x_-^2)\ln(x_-(1-x_-))\Bigg) \qquad \text{(B.17)} \\
& + \frac{Q^2}{16(pl)^2}\frac{Q^2(Q^2+2M^2)G_M^2 - 8M^4G_E^2}{(4M^2+Q^2)R(1,Q^2)} \\
& \times\Bigg(((1-x_-^2)\frac{(2+x_-)}{x_-} - 2\ln x_-)\ln\left(\frac{Q^2}{m^2}\right) + \frac{\pi^2}{3} - 2\mathrm{Li}_2(x_-) + \ln^2 x_- \\
& - (1-x_-^2)\frac{(2+x_-)}{x_-}\ln(1-x_-) - \left(\frac{2}{x_-} + 1 - 2x_- - x_-^2\right)\ln x_- \\
& - (1-x_-)(\frac{1}{x_-} + 5 + 2x_-)\Bigg) + \frac{Q^2}{4(pl)^2}\frac{Q^2G_M^2 - 2M^2G_E^2}{R(1,Q^2)}\ln x_-\Bigg].
\end{aligned}
$$

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