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TWO DIMENSIONAL GRAVITY FROM d = 0 AND d = 1 MATRIX MODEL

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Two Dimensional Gravity from

$d = 0$ and $d = 1$ Matrix Models

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ABSTRACT

The two nonperturbative formulations of *2d* gravity are compared. The first one is an analytical continuation of the matrix integral. This method provides a simple description of random surfaces statistics, but leads to complex expectation values. In the second method, proposed by Marinari and Parisi, observables in *2d* gravity are identified with the correlators for *1d* supersymmetric string. The correct quantization in the double scaling limit reduces the problem to calculation of a few eigenfunctions of simple one dimensional Hamiltonian. The function is proposed which may substitute the Painleve transcendent for the second definition of *2d* gravity. The universality of the model is also discussed.

1. INTRODUCTION

Since the double scaling limit [1 - 3] in the matrix models [4 - 6] was found the problem of nonperturbative investigation of two dimensional gravity attracts a permanent interest. In this limit it is easy to find the contribution to the partition function of surfaces with any fixed genus. But the sum over genera occurs to be the sum of asymptotic series and requires nonperturbative definition. At first sight the matrix integral itself provides the most natural nonperturbative definition of *2d* gravity. But this is not entirely so. For the pure gravity double scaling limit is realized if only the action in matrix integral is unbounded from below and thus the integral exists only in terms of perturbation theory.

Two methods are used for rigorous definition of a matrix integral. The first method is the analytical continuation over the coupling constant. The consequences of such analytical continuation are discussed in papers [7 - 9]. Due to nonperturbative corrections the partition function of 2d gravity becomes complex.

In this paper we would like to discuss in details a formulation of *2d* gravity via the *1d* supersymmetric string [10], or by the "fifth-time" formalism [11] (see also [12, 13]). The method of papers [10 - 13] leads naturally to real values of all observables. On the other hand all the practical calculations became now considerably more involved. Analytical results were obtained in the planar limit only [10]. For nonplanar contributions the authors of papers

[10, 11] have performed computer simulations with matrices of the size $N \sim 10$ (it is worthy to note that corrections to scaling are of order $N^{-1/5}$).

In section 2 of this paper we remind how the analytical continuation allows to fix nonperturbative features of $2d$ gravity.

The formulation of the method of papers [10 - 13] is given in section 3.

In section 4 we discuss the properties of the approximate solution of supersymmetric matrix Hamiltonian [10]. Expectation values which could be found with this approximate wave functions are identical to that found by analytical continuation of the matrix integral. But fermionic wave functions of sec. 4 are useful to compare two models of gravity.

In section 5 of this paper we consider a nonperturbative features of a model proposed by Marinari and Parisi [10]. The matrix quantum mechanics in double scaling limit is equivalent to quantum mechanics of fermions in universal bottomless potential (see, e.g., [12]). Crucial for the understanding of nonperturbative phenomena is the observation that in the double scaling limit the fermionic energy levels are quantized. All nonperturbative effects are saturated by a few eigenfunctions of simple one dimensional Hamiltonian.

In section 6 we compare the Green functions in both formulations of $2d$ gravity. The new expression for the Painleve transcendent was found. We also define the function which may substitute the Painleve transcendent for the second model of gravity.

In section 7 we examine the universality of the Marinari - Parisi model [10].

2. WHAT DO WE LEARN FROM THE ANALYTICAL CONTINUATION?

Partition function Z^a of $2d$ quantum gravity is determined by the integral over hermitian $N \times N$ matrices [4 - 6]

$$\exp(Z^a) = \int \exp[-N \operatorname{tr} V(M)] \, dM \, . \qquad (2.1)$$

The discretized random surfaces are associated with the dual Feynman graphs in the perturbative expansion of the integral (2.1). Birth of new handles is suppressed by the factor N^{-2}. The main result of papers [1 - 3] is the observation that in the double scaling limit $(y=(g_c-g)N^{4/5}$ fixed at $N\to\infty)$ "string susceptibility" $f(y)=Z''(y)$ satisfies the Painleve I equation

$$y = f^2 - f'' \, . \qquad (2.2)$$

Expansion over topologies now appears as asymptotic expansion for large y

$$f(y) = \sqrt{y} - \frac{1}{8y^2} - \frac{49}{128y^{9/2}} - \ldots.. \qquad (2.3)$$

All the information about $2d$ gravity which may be interpreted in terms of surfaces is stored in the coefficients of the asymptotic series (2.3). In order to obtain the full nonperturbative definition of $2d$ quantum gravity one has to specify uniquely the solution of Painleve equation. The large y behavior (2.3) fixes only one of boundary conditions for this solution.

As it was noted before, the scaling behavior described by eq. (2.2) takes place only for bottomless matrix actions $V(M)$ (2.1). Following the line of reasoning of [7, 8] one may consider as a nonperturbative definition of the quantum gravity not the integral (2.1) itself, but its analytical continuation from the point where the integral is well defined to point where it really corresponds to $2d$ gravity. The result of analytical continuation can be considered as the contour deformation for the integral over the matrix M eigenvalues x_i. The solution of the Painleve I equation which correspond to well defined complex matrix integral has the asymptotic:

$$f \underset{y\to\infty}{=} \sqrt{y} + \dots$$

$$f \underset{y\to-\infty}{=} i\sqrt{-y} + \dots \qquad (2.4)$$

David [7] has shown that in quantum gravity the so called "triply truncated solution" is realized. In paper [8] the asymptotic of the nonperturbative imaginary part of f was found for the particular action $V(M)$:

$$\operatorname*{Im}_{y\to\infty}(f) = \frac{\sqrt{3\sqrt{2}}}{4\sqrt{\pi}} \frac{1}{y^{1/8}} \exp\left\{-\frac{4\sqrt{2}}{5} y^{5/4}\right\} . \qquad (2.5)$$

The pre-exponential coefficient of Im(f) together with the boundary conditions (2.4) allows fix uniquely the solution of the Painleve I equation (2.2).

The price for the simple description of gravity by the analytical continuation of the integral (2.1) is imaginary nonperturbative part of correlators.

3. FORMULATION OF THE METHOD

Now we would like to discuss another very elegant definition of two dimensional gravity [10 - 13].

The authors of [10 - 13] proposed to interpret averaged values of operators in quantum gravity as the expectation values

$$\langle Q\rangle = \langle\Psi_0|Q|\Psi_0\rangle , \qquad (3.1)$$

over ground state wave function for quantum mechanical Hamiltonian

$$H = -\operatorname{tr}\left(\frac{\partial}{\partial M} - \frac{N}{2}V'\right)\left(\frac{\partial}{\partial M} + \frac{N}{2}V'\right) , \qquad (3.2)$$

where V is a polynomial action (2.1). The Hamiltonian (3.2) corresponds to purely bosonic sector of some supersymmetric matrix Hamiltonian.

The formal solution of the equation $H\Psi = 0$ is

$$\Psi^a = \frac{\exp\left\{-\frac{N}{2}\operatorname{tr} V(M)\right\}}{\sqrt{Z}} . \qquad (3.3)$$

For an action tr V(M) bounded from below the function Ψ^a is the exact ground state of H (3.2). But as pointed above there is no continuous limit in pure gravity for bounded actions. Nevertheless in this case the Hamiltonian (3.2) has a well defined ground state. The true eigenfunction Ψ_0 coincides with Ψ^a (3.3) to any order of perturbation theory. We associate random surfaces with perturbative expansion of (2.1) or (3.1). Therefore all the phenomena connected with surfaces in both formulations of $2d$ gravity coincide identically. Only nonperturbative corrections that cannot be expanded in powers of coupling constant differ for the two approaches.

This method to make unstable theories sensible was proposed in [14].

At the end of this paper we will return to the theories with arbitrary action $V(M)$ but now let us discuss the simplest model

$$V(M) = \frac{M^2}{2} - \frac{g}{3} M^3 . \qquad (3.4)$$

Hamiltonian H (3.2) in this case takes the form:

$$H = -\frac{\partial^2}{\partial M_{ij}\partial M_{ji}} + N^2 \operatorname{tr}\left\{\frac{(M - gM^2)^2}{4} + gM - \frac{1}{2}\right\} . \qquad (3.5)$$

The ground state wave function Ψ is a symmetric function of the eigenvalues x_i of the hermitian matrix M. It is useful to introduce the antisymmetric function $\Phi(x)$:

$$\Phi(x) = \prod_{i<j}(x_i - x_j)\,\Psi(x) = \det\left((x_i)^k\right)\Psi(x) . \qquad (3.6)$$

Now Φ is the wave function for the system of N noninteracting fermions [15]. The one fermion Hamiltonian is given by

$$H\,\varphi_i = \left\{ -\frac{\partial}{\partial x^2} + N^2\left[\frac{(x-gx^2)^2}{4} + gx - \frac{1}{2}\right]\right\} \varphi_i = E_i\varphi_i\,, \qquad (3.7)$$

$$\Phi(x) = \frac{1}{\sqrt{N!}} \det\left(\varphi_i x_j\right)\,.$$

4. PROPERTIES OF THE APPROXIMATE "SUPERSYMMETRIC" SOLUTION

Consider the function $\chi^k = x^k \exp\left[-\frac{N}{2} V(x)\right]$ - where $V(x)$ is the polynomial (3.4).

$$H\chi^k = -k\,(k-1)\,\chi^{k-2} + \frac{N(2k-N+1)}{2}\,\chi^k - (N-1-k)\,Ng\,\chi^{k+1}. \qquad (4.1)$$

Here H is the one particle operator (3.7). The first N functions χ^k $(0 \le k < N)$ form an invariant subspace for the operator H. That means that eq. (3.7) has N formal solutions

$$H\,\varphi_i^a = E_i^a\,\varphi_i^a\,, \qquad (4.2)$$

$$\varphi_i^a = p_i(x)\exp\left[-\frac{N}{2}V\right] = p_i(x)\exp\left[-\frac{N}{2}x^2 + \frac{Ng}{6}x^3\right]$$

where $p_i(x)$ is a polynomial of the power N-1 . Moreover,

$$\sum_{i=0}^{N-1} E_i^a \equiv 0\,. \qquad (4.3)$$

The Hamiltonian H (3.7) resembles the so called quasi exactly solvable Hamiltonians (see [16]). The N solutions of eq. (3.7) can be found algebraically. But φ_i^a (4.2) grows to infinity at $x \to \infty$ and therefore is not the true eigenfunction of H.

The polynomials p_i (4.2) should not be confused with orthogonal polynomials P_n which are usually used in random matrix theory (see e.g. [1]). The polynomials $P_n = x^n + \dots$ are of the power n , but they are not solutions of any differential equation. All the polynomials p_i are of the same power N-1 and they are solutions of the differential equation (4.2).

Like Ψ^a (3.3) for not too large g is an approximate eigenfunction of Hamiltonian H (3.5), the functions φ_i^a are the approximate eigenfunctions of one particle operator H (3.7). φ_i^a and E_i^a coincide with the exact eigenfunction and energy φ_i, E_i to any order of the coupling g.

We can calculate the expectation values of any operator in $2d$ gravity with functions φ_i^a provided that the integrals are taken along the contour $(-\infty,\ \exp(i\pi/3)\cdot\infty)$. Moreover in this case

$$\int_{-\infty}^{\exp(i\pi/3)\cdot\infty} \varphi_i^a\,\varphi_j^a\,dx \sim \delta_{ij}\,. \qquad (4.4)$$

Such a definition of $2d$ gravity is identical to the analytical continuation over coupling constant in the integral (2.1) described in [7, 8].

Let us write the polynomials $p_i(x)$ in the form

$$p_i = \prod_{k=1}^{N-1} (x - x_k), \qquad (4.5)$$

where x_k are zeros of the polynomial, not necessarily real. From (4.2) we get:

$$-\sum_{\substack{n,k \\ n\neq k}} \frac{1}{(x-x_k)(x-x_n)} + N\sum_k \left(\frac{x-gx^2}{x-x_k} + g\,x - \frac{1}{2}\right) = E_i\,. \qquad (4.6)$$

The absence of poles in (4.6) l.h.s. implies the system of equations:

$$-\sum_{n\neq k}\frac{2}{x_n - x_k} + N\,(x_k - g\,x_k^2) = 0\,. \qquad (4.7)$$

This system allows a simple physical interpretation. The equations (4.7) are the equilibrium conditions for N-1 particles moving in the external field $V = N\,(x^2/2 - gx^3/3)$ and repulsing each other with the potential $U_{ij} = -\log(x_i - x_j)^2$. The equilibrium distribution of such particles have been discussed in [15]. In the method of paper [15] this distribution describes the saddle point of the integral over matrix eigenvalues. But the saddle point approximation can be applied only at $N \to \infty$. In our approach (4.7) are the exact equations for zeros of the polynomials $p_i(x)$.

Equations (4.7) have N different solutions corresponding to different solutions of (4.2). At $g < g_c$ only one of the solutions of (4.7) (that of the maximum energy E^a_{N-1}) corresponds to the stable distribution of N-1 repulsing particles. The second solution corresponds to unstable equilibrium distribution of particles. In all solutions with lower energies E^a_n some of the zeros x_k became complex. At $g < g_c$ all E^a_i are real. The zeros of polynomials p_i with largest numbers on the complex plane are shown schematically on fig. 1. Also we have shown on the figure the averaged one particle field V_{eff} which is the external field (3.4) plus repulsion from another particles averaged over the equilibrium distribution.

At some point $g \approx g_c$ the equilibrium distribution of N-1 particles with real coordinates proves to be impossible. After that point some of the eigenvalues E^a became also complex.

5. DOUBLE SCALING LIMIT

Hamiltonian H (3.7) at small g describes the fermions in the potential having two minima. Fermi energy in WKB approximation is defined by the integral

$$\int\sqrt{E_F - U(x)}\;dx = N\pi\,. \qquad (5.1)$$

The authors of [10] have shown that E_F coincides exactly with the bottom of the second well (the local minimum)

$$E_F \equiv U_{min}\,. \qquad (5.2)$$

Let us introduce the new scaling variables y, λ and ε instead of x, g and E (3.7) [12]

$$g = \frac{1}{2\cdot 3^{3/4}} - \frac{1}{N^{4/5}}\,\frac{\lambda}{4\cdot 2^{1/5}\cdot 3^{3/20}}\,,$$

$$x = \frac{3+\sqrt{3}}{6g} + \frac{1}{N^{2/5}}\,2^{2/5}\cdot 3^{1/20}\,y\,, \qquad (5.3)$$

$$E = N^2\left(\frac{1}{144g^2} + \frac{\sqrt{3}}{6}\right) + N^{4/5}\,\frac{\varepsilon}{2^{4/5}3^{1/10}}\,.$$

The Hamiltonian (3.7) now takes the form

$$H = -\frac{\partial^2}{\partial y^2} - \lambda y + \frac{y^3}{3} + \frac{1}{2^{8/5}\cdot 3^{6/5}}\,\frac{y^4}{N^{2/5}}\,. \qquad (5.4)$$

Last term here formally goes to zero at $N \to \infty$, but it is just the term which determines the phase of wave functions at $y \to -\infty$. At $\lambda > 0$ it is convenient to shift the energy and coordinate

$$y = \sqrt{\lambda} + z\,;\; \varepsilon = E + \frac{2}{3}\lambda^{3/2}\,,$$

$$h = -\frac{\partial^2}{\partial z^2} + U(z) = \qquad (5.5)$$

$$= -\frac{\partial^2}{\partial z^2} + \omega^2 z^2 + \frac{z^3}{3} + \frac{1}{2^{8/5}\cdot 3^{6/5}}\,\frac{z^4}{N^{2/5}}\,,$$

where $\omega = \lambda^{1/4}$.

The semiclassical result of [10] - (5.1, 5.2) now can be written as:

$$E_F \equiv 0 \; ; \int p\, dz = \int \sqrt{-U(z)}\, dz = N\pi \, . \qquad (5.6)$$

The following step is the calculation of individual energy levels. The semiclassical values are given by the Bohr's quantization rule:

$$\int p_n\, dz = \int \sqrt{E_n - U(z)}\, dz = \left(n + \frac{1}{2}\right)\pi \, , \qquad (5.7)$$

$$n = 0,\ 1, \ldots, N-1 \, .$$

This formula can be applied for upper levels if only $\lambda >> 1$ ($\omega >> 1$). In this case the energy interval $E_n - E_{n-1}$ is a slow function of number

$$\left(\frac{\partial E_n}{\partial n}\right)^{-1} = \frac{1}{2\pi}\int \frac{dz}{p} = \frac{T}{4\pi} \, , \qquad (5.8)$$

where T is the period of motion over the classical orbit. The nice feature of Hamiltonian (5.5) is that while the orbit at $N \to \infty$ spreads to infinity, the period (twice the time to escape to infinity) proves to be finite. From (5.8) one easily gets

$$\frac{\partial E_n}{\partial n} = 2\omega \, . \qquad (5.9)$$

We have got the unexpected result - energy levels in bottomless potential $U = \omega^2 z^2 + z^3/3$ are quantized. Expression (5.9) shows the interval between nearest levels. An absolute position of a level depends on the way we define U at $z \to -\infty$. That is why we keep the term $\sim z^4/N^{2/5}$ in the Hamiltonians (5.4, 5.5). Comparing (5.6), (5.7), (5.9) one easily gets.

$$E_{N-1} = -\,\omega,$$

$$E_{N-2} = -\,3\omega \, , \qquad (5.10)$$

$$E_{N-3} = -\,5\omega.$$

. . . .

These expressions are correct at $\omega >> 1$ and $|N-n| << \omega^5$. As we shall see below at $\omega >> 1$ all the interesting nonperturbative effects are sensitive to behavior of the upper ($N-1$) - th fermion only.

At $\lambda \sim \omega \sim 1$ semiclassical expression (5.10) fails. Nevertheless in this case (just like at $\lambda < 0$) the energy levels are quantized $E_{n+1} - E_n \sim 1$.

At large negative z the eigenfunctions of Hamiltonian (5.5) take the form:

$$\varphi \sim \frac{1}{|z|^{3/4}} \cos\left\{\frac{2}{5\sqrt{3}}(a-z)^{5/2} - \frac{2a}{3\sqrt{3}}(a-z)^{3/2} - \frac{\sqrt{3E}}{\sqrt{-z}} + \frac{\pi}{4}\right\} \, , \qquad (5.11)$$

where $a = -3\omega^2$ is the classical turning point at $E = 0$. The expression (5.11) is correct at $-z >> 1$, $-z >> \omega^2$ for any ω. We see what at $z \to -\infty$ all the wave functions have the same phase.

The density of fermions is given by

$$\rho(z) = \sum \varphi_i^2\,(z) \, . \qquad (5.12)$$

Within the semiclassical limit at $-z >> 1$

$$\rho(z) \approx \frac{|z|\sqrt{a-z}}{\pi\sqrt{3}} - \frac{1}{4\pi|z|} \cos\left\{\frac{4}{5\sqrt{3}}(a-z)^{5/2} - \frac{4a}{3\sqrt{3}}(a-z)^{3/2}\right\} \, ,$$

$$a = -3\omega^2 \, . \qquad (5.13)$$

Here the first term is the usual semiclassical density found in the early paper [15]. The second oscillating term appears due to wave functions coherence (5.11).

The explicit form of fermionic density may be found numerically. Formula (5.11) gives the exact boundary condition for φ_i at $z \to -\infty$. The function $\rho(z)$ at $g = g_c$ (i.e. $\lambda = \omega = 0$) is presented on the fig. 2. The $\rho(z)$ was found as a sum over 40 upper eigenfunctions of the Hamiltonian (5.5). The small oscillations which appear due to coherence of φ_i (5.11) are seen quite clearly on the figure. The smooth part of $\rho(z)$ is well approximated by the semiclassical expression $\rho(z) = (-z)^{3/2}/\pi/\sqrt{3}$.

All the properties of two dimensional surfaces are encoded in the coefficients of the asymptotic series (2.3). The only problem is the calculation of nonperturbative corrections which cannot be interpreted in terms of surfaces. In order to fix such effects it appears to be useful to calculate only the difference between values of observables for two approaches to 2*d* gravity.

Let us calculate, for example, the ground state energy $E = \sum E_i$ (3.7) of the Hamiltonian H (3.5). The value of E vanishes to any order of the coupling constant. It becomes to be nonzero due to nonperturbative supersymmetry violation. In section 4 we have introduced the polynomial solutions φ_i^a(4.2) of eq. (3.7). The functions φ_i^a are not eigenfunctions of the Hamiltonian H (3.7), but the energies E_i^a satisfy the identity $E^a = \sum E_i^a \equiv 0$. It is important that solutions φ_i^a decrease not at $x \to +\infty$, but at $x \to i\infty$. In the double scaling limit there exist two types of boundary conditions for Hamiltonian h (5.5) eigenfunctions.The first type is: $\varphi_i \to 0$ at $z \to \infty$ and φ_i behave like (5.11) at $z \to -\infty$. That very wave functions determine the true averages in 2*d* gravity. The second type is: φ_i^a behave like (5.11) at $z \to -\infty$, but $\varphi_i^a \to 0$ at $z \to i\infty$. The energies of just these wave functions E_i^a satisfy the identity (4.3). We are to calculate only the difference $E_i - E_i^a$ which determines the supersymmetry violation.

At $\omega \gg 1$ the expression for $E_i - E_i^a$ may be found explicitly. In this case the supersymmetry violation proves to be sensitive only to the upper occupied level behavior.

At $\omega \gg 1$ the wave function φ_{N-1} is well described by the WKB approximation everywhere except for $z \approx a \approx -3\omega^2$ (the classical turning point) and $z \approx 0$ (the motion close to the local minimum of the potential (5.5)). In the vicinity of $z = 0$ φ is a solution of the Schrödinger equation (5.5, 5.10)

$$\left(- \frac{\partial^2}{\partial z^2} + \omega^2 z^2 \right) \varphi = - \omega \varphi . \qquad (5.14)$$

Solutions of this equation φ_{N-1} and φ_{N-1}^a are:

$$\varphi_{N-1}^a = \exp\left(\frac{\omega z^2}{2} \right) ,$$

$$\varphi_{N-1} = \exp\left(\frac{\omega z^2}{2} \right) \sqrt{\frac{\omega}{\pi}} \int_z^\infty \exp\left(- \omega y^2 \right) dy . \qquad (5.15)$$

The first solution decreases at $z \to i\infty$, the second at $z \to +\infty$. At $z \ll -1/\sqrt{\omega}$

$$\varphi_{N-1} = \varphi_{N-1}^a - \frac{\exp\left(- \frac{\omega z^2}{2} \right)}{2\sqrt{\pi\omega}\,(-z)} . \qquad (5.16)$$

On the other hand, near the classical turning point $a \approx \approx -3\omega^2$

$$\varphi \sim \frac{1}{\sqrt{p}} \cos\left(\int_z^a |p|\, dx - \frac{\pi}{4} + \theta \right) \qquad (5.17)$$

at $z < a$ and

$$\varphi \sim \frac{1}{2\sqrt{p}} \exp\left(-\int_a^z |p|\, dx\right) + \frac{\theta}{2\sqrt{p}} \exp\left(\int_a^z |p| dx\right) \qquad (5.18)$$

at $z > a$. We suppose that $\sin\theta \approx \theta$. Sewing together (5.16) and (5.18) at $a << z << -1/\sqrt{\omega}$ one may found the phase shift $\theta - \theta^a$. The slight modification of the quantization condition (5.7) combined with (5.9) leads to:

$$E_{N-1} - E^a_{N-1} = -\frac{2\omega}{\pi}(\theta - \theta^a) . \qquad (5.19)$$

The final result is given by

$$E_{N-1} - E^a_{N-1} = \frac{\exp\left(-\frac{24}{5}\omega^5\right)}{12\pi^{3/2}\omega^{3/2}} . \qquad (5.20)$$

Very similar calculation shows that for the second level

$$E_{N-2} - E^a_{N-2} \sim \frac{\left(E_{N-1} - E^a_{n-1}\right)}{\omega^5} . \qquad (5.21)$$

Thus the supersymmetry violation far from the critical point is saturated by the upper occupied level.The ground state energy of the Hamiltonian H (3.5) below the phase transition point is (see (5.3))

$$E = N^{4/5} \frac{\exp\left\{-\frac{96}{5} 2^{3/4} 3^{3/16} N(g_c - g)^{5/4}\right\}}{24\pi^{3/2} 2^{5/8} 3^{5/32}[N(g_c - g)^{5/4}]^{3/10}} . \qquad (5.22)$$

Near the critical point ($\omega \sim \lambda \sim 1$) the value of E can be found numerically. Results of numerical calculation of a few upper energy levels $E_k(\lambda)$ and $E^a_k(\lambda)$ are shown on fig. 3. In order to find φ^a_k, E^a_k we solve the Schrödinger equation with complex boundary conditions. Therefore the values of E^a_k

are not necessary real. At some, small enough, value of λ two eigenvalues E^a_k and E^a_{k-1} merge and became complex for smaller λ.

6. THE GREEN FUNCTIONS IN 2*d* GRAVITY

The correlators for 2*d* gravity can be found easily if the model is defined by analytical continuation. One may add source term $J\cdot \mathrm{tr}(M)$ to the matrix action. The Green functions can be found as derivative over J of the partition function Z^a (2.1). In the double scaling limit all the Green functions are expressed through the derivatives of the universal solution of Painleve equation (2.2) f with boundary conditions (2.4, 2.5) (see e.g. [17, 18])

$$\left(\frac{\partial^n Z^a}{\partial J^n}\right)_{J=0} = \langle(\mathrm{tr}\, M)^n\rangle_c = -\left(-\frac{\alpha}{N^{1/5}}\right)^n \frac{f^{(n-2)}(x_c)}{6} . \qquad (6.1)$$

Here the subscript c means the connected part of the average. The only parameter α in (6.1) depends on the choice of the matrix action $V(M)$. For the model (3.4)

$$\alpha = 2^{3/5}\cdot 3^{9/20} ,$$

$$x_c = 4\cdot 2^{3/5}\cdot 3^{19/20}\cdot N^{4/5}\cdot(g_c - g) . \qquad (6.2)$$

The first correlator in (6.1) is

$$\frac{\partial Z^a}{\partial J} = -\frac{\alpha}{N^{1/5}} \int_{x_c} \frac{f(x)}{6} dx . \qquad (6.3)$$

The integral diverges at large x (see (2.3)). In terms of surfaces this divergence reflects the fact that Green function $\partial Z/\partial J$ is dominated by very small flat surfaces consisting of only a few triangles. Small flat surfaces also give an infinite model dependent contribution to the second correlator $\partial^2 Z/\partial J^2$. Nevertheless the contribution of surfaces which have at least one handle and nonperturbative corrections to these correlators are well defined in the limit of continuum theory. The simplest Green function which is well defined in continuous $2d$ gravity is $\partial^3 Z/\partial J^3$. By the way this correlator may be considered as an amplitude of a real process - decay of one string into two.

For gravity defined by the method of Marinari and Parisi the first correlator is given by

$$\langle \mathrm{tr}M\rangle = \sum_{i=0}^{N-1} \frac{\langle i|x|i\rangle}{\langle i\,|\,i\rangle} . \tag{6.4}$$

Here bra and ket vectors $\langle i|$, $|i\rangle$ are the single fermion states (3.7). In terms of scaling variables (5.3)

$$\langle \mathrm{tr}M\rangle = \sum \left\{ \frac{3+\sqrt{3}}{6g} + \frac{2^{2/5}3^{1/20}}{N^{2/5}} \frac{\langle i|y|i\rangle}{\langle i\,|\,i\rangle} \right\} . \tag{6.5}$$

The second term here is formally small. But all the information about large surfaces, topology and nonperturbative effects is contained in the term $\sim N^{-2/5}$ (6.5).

Fermionic wave functions at $y \to -\infty$ decrease like $|y|^{-3/4}$. The average value $\langle i|y|i\rangle$ comes from $y \sim -N^{1/5}$. All the wave functions at $y \to -\infty$ differ only by the normalization constant (5.11) and the value of $\langle i|y|i\rangle$ may be found in the WKB approximation. Thus the singular part of (6.5) is given by:

$$\langle \mathrm{tr}M\rangle = \frac{-\pi}{N^{1/5}}\, 2^{1/5}\, 3^{13/20} \sum^{N-1} \left(\int \varphi_i^{\,2} dy \right)^{-1} . \tag{6.6}$$

Here the wave functions φ_i are normalized by the condition (see (5.4, 5.11))

$$\varphi_i \underset{y\to-\infty}{\to} \frac{1}{(\varepsilon - U)^{1/4}} \cos\left(\Theta + \frac{\pi}{4} \right) . \tag{6.7}$$

$$\Theta = \int_{-\infty}^{y} \sqrt{\varepsilon - U} - N\pi = \int_{\infty}^{y} \left(\sqrt{\varepsilon-U} - \sqrt{-U} \right) + \int_{y}^{a} \sqrt{-U} .$$

This formula provides one of the boundary conditions for the functions φ_i. As we have seen before all the difference of the two formulations for $2d$ gravity reduces to the different choice of the second boundary condition for the Hamiltonian h (5.4) eigenfunctions. The analytical continuation of the matrix integral is associated with the condition $\varphi_i^a \to 0$ at $y \to i\infty$. In this case $\langle \mathrm{tr}M\rangle^a$ (6.6) should coincide with (6.3). Thus we have found the new formula for the Painleve transcendent which is realized in $2d$ gravity:

$$\int_{x_c} f(x)\, dx = \pi\, 18^{3/5} \sum \left(\int \varphi_i^{a\,2} dy \right)^{-1} , \tag{6.8}$$

$$\lambda = x_c\, 18^{-2/5} .$$

Here $f(x)$ is the solution of Painleve equation (2.2) with boundary conditions (2.4, 2.5). Right hand side of (6.8) is a function of λ (5.3, 5.4). Both right and left hand sides of (6.8) diverge. But the second derivative of (6.8) over x_c is well defined.

Formula (5.8) allows to calculate the sum in (6.8) at $\lambda \gg 1$. Up to nonsingular constant r.h.s of (6.8) is $18^{3/5}\varepsilon_F = -18^{3/5}2/3\lambda^{3/2} = -2/3\, x_c^{3/2}$ (compare with (2.3)). We have also calculated the imaginary part of (6.8) at x_c, $\lambda \gg 1$.

Expression (6.8) allows us to define the new function $\tilde{f}$ which may substitute the Painleve transcendent in the new definition of $2d$ gravity. One should only change the boundary condition for φ_i ($\varphi_i \to 0$ at $y \to \infty$ instead of $\varphi_i^a \to 0$ at $y \to i\infty$). It seems very attractive to get all the Green functions for gravity defined via Marinari and Parisi suggestion by the formula (6.1) with $\tilde{f}$ used instead of f. As we have seen (6.6) this is really so for the simplest correlator <tr M>.

For new formulation of gravity one may also add source term $J \cdot \mathrm{tr}(M)$ to the matrix action. The matrix Hamiltonian (3.2) also should be complemented with source term. The formulas like (6.1) allow one to define only the correlators, not the partition function. One may consider the formula

$$\frac{\partial Z}{\partial J} = \langle \mathrm{tr}\, M \rangle , \qquad (6.9)$$

as a definition of the partition function for $2d$ gravity. Now all the Green functions should be given by (6.1) with f replaced by $\tilde{f}$. Unfortunately we do not know whether the Green functions defined as derivatives of $\tilde{f}$ are the connected averages of any operator.

7. UNIVERSALITY

Up to now we have discussed only the model where random surfaces are generated by the simplest action (3.4) (the surfaces are glued from triangles). For gravity defined by the analytical continuation of integral (2.1) all the observables in double scaling limit do not depend on the specific choice of the matrix action $V(M)$ [7]. It is interesting to understand whether such an universality takes place in gravity defined via the Marinari and Parisi suggestion [10].

Let us for example instead of (3.4) discuss the action with stronger anharmonicity

$$V = \frac{M^2}{2} - \frac{g}{5} M^5 . \qquad (7.1)$$

The generalization for arbitrary $V(M)$ should be evident. Now the fermionic Hamiltonian (3.7) takes the form:

$$H = \sum_i \left\{ - \frac{\partial^2}{\partial x_i^2} + N^2 \left[\frac{(x_i - g x_i^4)^2}{4} + g x_i^3 - \frac{1}{2} \right] \right\} + \qquad (7.2)$$
$$+ Ng \left(\sum_i x_i^2 \right) \left(\sum_j x_j \right).$$

We are faced with the problem of interacting fermions. The natural way to handle this problem is the mean field approximation [13]. For the specific action (5.21) the fermion - fermion interaction leads to an additional term in single particle effective potential at high N

$$\Delta U_{eff} = N^2 (Ax + Bx^2) , \qquad (7.3)$$

where $A, B \sim 1$.

At $g < g_c$ the semiclassical fermionic density may be found by the method of paper [15]. The authors of [15] have used the steepest descent method to calculate the integral over the matrix eigenvalues. Saddle point for this integral formally coincides with the equilibrium configuration of N particles repulsing with the potential $U_{ij} = -\log(x_i - x_j)^2$ in the external field (7.1). At large N it is natural to introduce the density of eigenvalues $\rho(x)$. For the action (7.1) the equilibrium density is given by

$$\rho(x) = \frac{N}{2\pi} (\alpha + \beta x + \gamma x^2 - g x^3) \sqrt{(b + x)(a - x)} . \qquad (7.4)$$

The constants α, β, γ, b, a can be found algebraically.

In the matrix quantum mechanics the density of eigenvalues $\rho(x)$ should be interpreted as a semiclassical fermionic density:

$$\rho(x) = \frac{p_F(x)}{\pi} = \frac{1}{\pi} \sqrt{E_F - U_{eff}} . \qquad (7.5)$$

Here E_F, p_F are the Fermi energy and Fermi momentum. The points a, $-b$ are the classical turning points for upper fermion level.

The function $p_F(x)$ has a zero at some point $x_{min} > a$. That means that the effective potential U_{eff} has two minima (at $g<g_c$) and the Fermi energy E_F at large N equals to U_{min} - the local minimum of U_{eff}. Instead of (5.1, 5.2) for arbitrary $V(M)$ we get

$$\int \sqrt{U_{min} - U_{eff}(x)}\, dx = N\pi + \delta(g) \qquad (7.6)$$

where $\delta \sim 1$ is a small correction to WKB result. The expression (7.6) is correct for any value of $g < g_c$. Of course $\delta = \delta\,(g)$ is a function of the coupling. But there are now reasons for δ to varies fast with $\lambda \sim (g - g_c)N^{4/5}$.

The double scaling limit corresponds to the motion of the fermions in a potential (compare with (5.5))

$$U = \omega^2 z^2 + \frac{z^3}{3} + \frac{z^4}{N^{2/5}}\, F\left(\frac{z}{N^{2/5}}\right) . \qquad (7.7)$$

Here all the information about the action V(M) is stored in the function F. Instead of (7.6) one has:

$$\int p\, dz = \int \sqrt{-U(z)}\, dz = N\pi + \delta(g_c) , \qquad (7.8)$$

and $\delta(g_c)$ is a constant ~ 1. In order to fix the phase of wave functions at $z \to -\infty$ and to quantize the energy it is enough to know the value of $\delta(g_c)$. In the model with the action $V = M^2/2 + gM^3/3$ (3.4) we had $\delta \equiv 0$ (5.6). To prove the universality it's enough to prove that $\delta(g_c) = 0$ for any matrix action. We know that at large enough ω the fermionic density differs very slightly, by the correction of the order $\sim \exp(-c\omega)^5$, from the universal density of eigenvalues of the integral (2.1) defined by the analytical continuation. If $\delta(g_c)$ in (7.8) really doesn't depend on ω then for arbitrary $V(M)$ there must be $\delta(g_c) = 0$.

Thus the only, very natural, assumption that phase shift $\delta(g)$ (7.6) does not depend on the scaling variable $\lambda \sim$ $\sim (g_c - g)N^{4/5}$ seems to be enough to prove the universality of Marinari and Parisi model.

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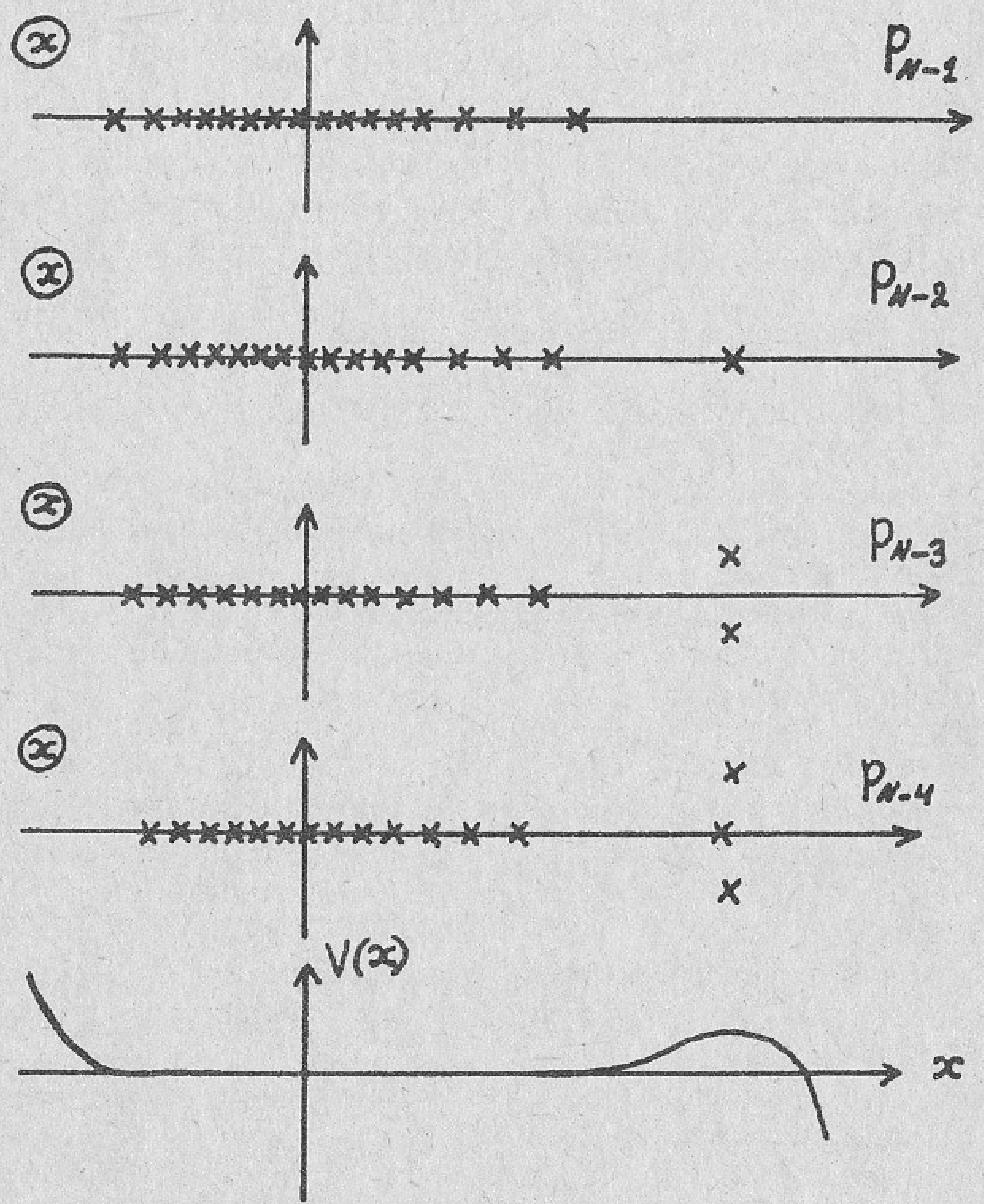

Fig. 1. The zeros on the complex plane for polynomials p_{N-1}, p_{N-2}, p_{N-3} and p_{N-4}. The effective one particle potential $V_{\mathrm{eff}}(x)$ is the external field $V(x)$ (3.4) plus the averaged repulsion of the zeros $< -\log(x_i - x_j)^2 >$.

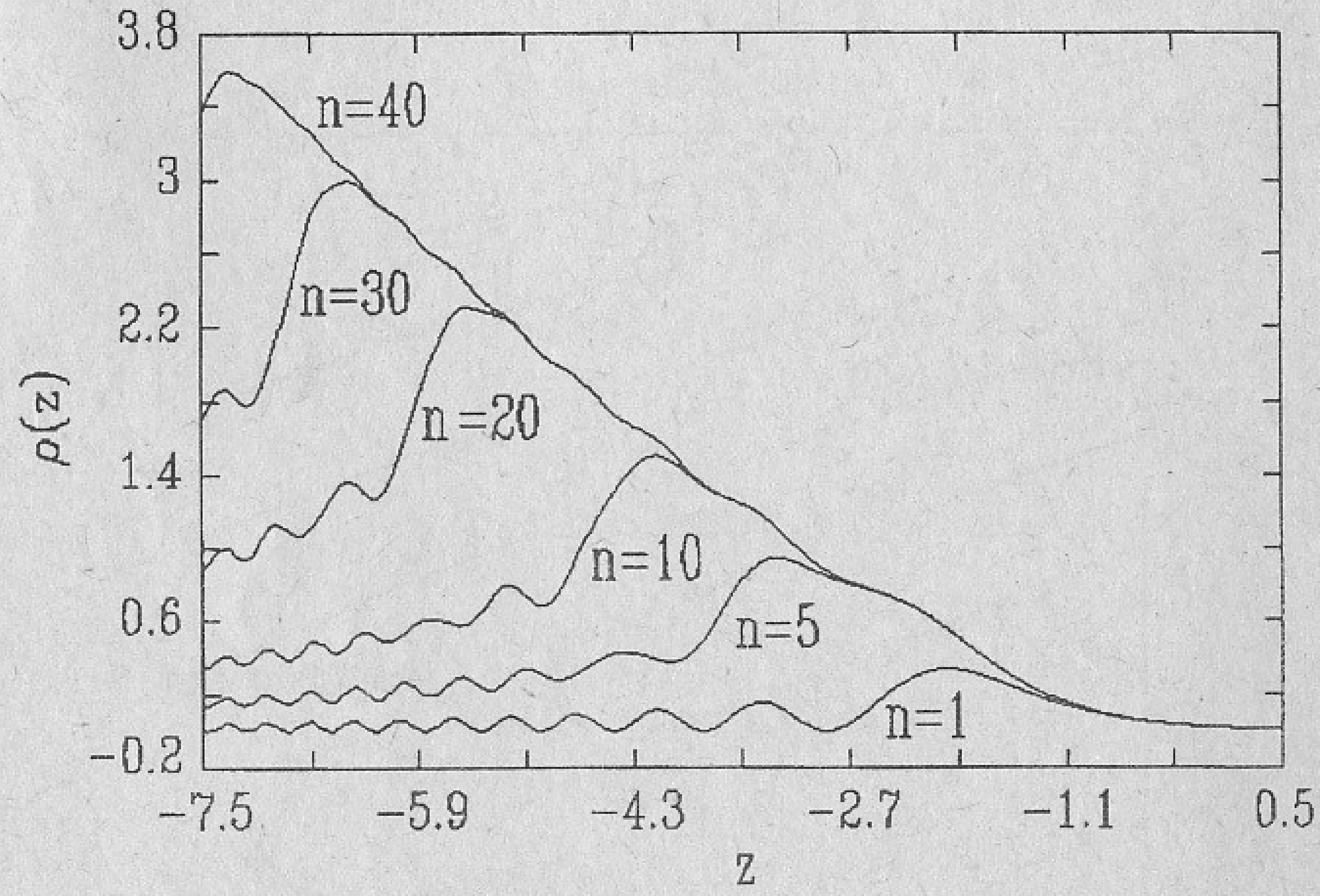

Fig. 2. The density of fermions at $g = g_c$. The density was found as a sum over 1, 5, 10, 20, 30, and 40 upper fermions.

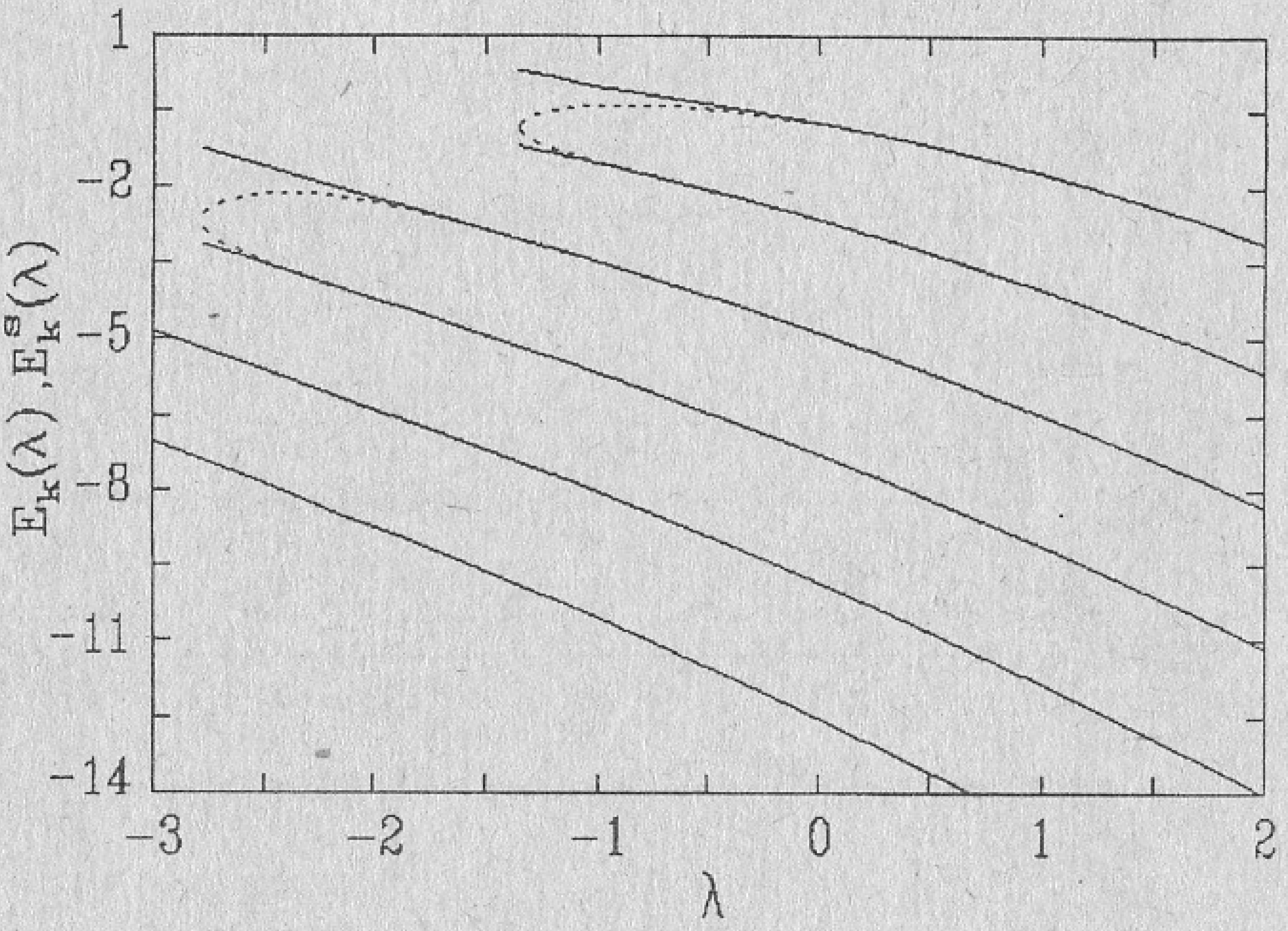

Fig. 3. The six upper energy levels $E_k(\lambda)$ and $E_k^a(\lambda)$. Solid line is $E_k(\lambda)$ – the energy levels for solutions with usual boundary condition $\varphi_k \to 0$ at $y \to \infty$. Dotted line is $E_k^a(\lambda)$ – the energy levels for solutions satisfying the complex boundary condition $\varphi_k^a \to 0$ at $y \to i\infty$.

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П.Г. Сильвестров

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