26

ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

A.D. Dolgov, S.V. Kostyuk, I.B. Khriplovich

ON THE PHOTONIC CHIRAL ANOMALY IN AN EXTERNAL GRAVITATIONAL FIELD

PREPRINT 91-31

НОВОСИБИРСК

1. The anomaly of the photonic chiral current in an external gravitational field was found independently in Refs. [1, 2]. In Ref. [1] it was obtained by means of the infrared regularization in dispersion-relation approach of Ref. [3] and was formulated as

$$\langle \nabla_\mu K^\mu \rangle = - \frac{1}{96\pi^2} R_{\mu\nu\alpha\beta} \tilde{R}^{\mu\nu\alpha\beta} , \qquad (1)$$

where $K^\mu = - \frac{1}{\sqrt{-g}} \varepsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta$ is the photonic chiral current, Δ_μ is a covariant derivative, brackets <...> denote the expectation value over the photon vacuum in an external gra- vitational field, $R_{\mu\nu\alpha\beta}$ is the Riemann tensor.

Anomaly (1) is not the first bosonic chiral anomaly in the four-dimensional space discussed in literature. In Ref. [4] the chiral anomaly of an antisymmetric tensor field in an external gravitational one was considered. This anomaly however cancels out completely by the anomaly of the vector ghosts [5], their introduction being necessary for the tensor field quantization. This result is quite natural from the physical point of view since a massless antisymmetric tensor field on-mass-shell is equivalent to a scalar one. And the latter cannot possess neither anomaly, nor the chiral current itself.

One more bosonic chiral anomaly discussed in literature [6] refers to gluons in a gluon background. However, as it has been demonstrated by a direct calculation in Ref. [7], the corresponding triangle diagram contains logarithms, so here as well the situation in essence is not anomalous (see also Ref. [8]).

This fact as well has a simple physical explanation. It is convenient to start from the well-known case of the Adler anomaly of the fermion axial current in an external vector field. Let us consider, following Ref. [3], the imaginary part of Fig. 1. Its left block corresponds to the creation by the scalar operator $\partial_\mu a^\mu$ of a pair, massless fermion and antifermion. Due to the angular momentum conservation, the fermion and antifermion are of the same chiralities in the centre-of-mass frame. But then the right block of the diagram turns to zero by a dynamical reason: massless fermion and antifermion of the same chirality do not annihilate. The exception is the point $p^2 = 0$, p being the total momentum, where there is no centre-of-mass frame, particle and antiparticle have

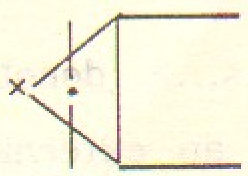

Fig. 1.

parallel momenta and, due to the same angular momentum conservation, opposite chiralities. In result the imaginary part of the amplitude, divided by p^2, turns out proportional to $\delta(p^2)$ which leads to the anomaly in the amplitude itself (for more detailed discussion see Ref. [3]).

However, when we deal with internal Yang - Mills quanta of the same chirality, there are no reasons preventing their transition into external quanta, i.e., their rescattering. Therefore, the imaginary part of the triangle diagram differs from zero at all $p^2 > 0$. As a result, the amplitude turns out in essence nonanomalous one. (However, it has certainly an anomalous contribution as well which is due to the spin-flip rescattering of the quanta whose initial chiralities were of the opposite signs. Just this, anomalous part of the amplitude was calculated in Ref. [6].)

Just the same arguments demonstrate that the case of the gravitons in an external gravitational field as well is nonanomalous one. Thus, if one abstracts from the exotic case of higher spin theories, a photon (or gluon) in an external gravitational field is the only clean case of the bosonic chiral anomaly in the four-dimensional space.

2. Due to this fact an alternative derivation of this anomaly is of a certain interest. Previously relation (1) was confirmed in various ways in Refs. [9, 10]. Here we wish to present the derivation of the photonic chiral anomaly by means of the customary Pauli - Villars regularization. To

simplify the calculations we use the technique of Green's functions in an external field.

The Lagrangian of photons in an external gravitational field is in the Feynman gauge

$$\mathcal{L} = \frac{1}{2} A^{\mu} \Box A_{\mu} , \tag{2}$$

where $\Box = \nabla_{\mu} \nabla^{\mu}$ is the covariant Dalembert operator. The term with the Ricci tensor $R_{\mu\nu} = R_{\mu\alpha\nu\beta} g^{\alpha\beta}$ is omitted here and below since it certainly does not contribute to the anomaly. Neither do we consider the scalar ghosts that also do not contribute to the chiral current and its anomaly.

The chiral photonic current is constructed as

$$K^{\mu}(y) = \lim_{x \to y} - \frac{1}{\sqrt{-g}} \varepsilon^{\mu\nu\alpha\beta} A_{\nu}(x) \partial_{\alpha} A_{\beta}(y) . \tag{3}$$

It has been shown in Refs. [10, 11] that the chirality of photons is conserved in an external gravitational field. It would lead to the vanishing of the vacuum expectation value in a gravitational field of the operator: $\nabla_{\mu} K^{\mu}$:

$$\langle \nabla_{\mu} K^{\mu} \rangle = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \langle K^{\mu}(x) \rangle = \lim_{x \to y} - \frac{1}{\sqrt{-g}} \varepsilon^{\mu\nu\alpha\beta} \langle A_{\nu,\mu}(y) \times$$

$$\times A_{\beta,\alpha}(x) \rangle , \qquad A_{\mu,\nu}(x) \equiv \frac{\partial}{\partial x^{\nu}} A_{\mu}(x) . \tag{4}$$

However, the current $K^{\mu}(x)$ is a singular one and should be regularized. According to the Pauli - Villars prescription, we introduce the regulator vector field with the mass M (it will finally tend to infinity) and the Lagrangian

$$\mathcal{L}^{M} = \frac{1}{2} \tilde{A}_{\mu} \Box \tilde{A}^{\mu} + \frac{1}{2} M^{2} \tilde{A}_{\mu} \tilde{A}^{\mu} . \tag{5}$$

The regularized current $K^{\mu}_{reg}(x)$ is

$$K^{\mu}_{reg}(x) = K^{\mu}(x) - K^{\mu}_{M}(x) , \tag{6}$$

where $K^{\mu}_{M}(x)$ is the chiral current of the regulator field $\tilde{A}_{\mu}$. The expectation value of interest to us of the regularized chiral current in an external gravitational field is

$$\langle \nabla_{\mu} K^{\mu}_{reg}(x) \rangle = - \langle \nabla_{\mu} K^{\mu}_{M}(x) \rangle =$$

$$- \lim_{x \to y} \left(- \frac{\varepsilon^{\mu\nu\alpha\beta}}{\sqrt{-g}} \right) \frac{\partial}{\partial x^{\mu}} \frac{\partial}{\partial y^{\alpha}} D_{\nu\beta}(x, y) . \tag{7}$$

Here $D_{\nu\alpha}(x, y)$ is the Green function of the field $\tilde{A}_{\mu}$ which satisfies the equation

$$(\Box + M^{2}) D_{\nu\alpha}(x, y) = i g_{\nu\alpha}(x) \delta^{4}(x - y) . \tag{8}$$

$D_{\nu\alpha}(x, y)$ is a bivector, all the derivatives in eq. (8) act on the coordinate x and on the first index of this bivector.

We shall solve eq. (8) by expansion in powers of the gravitational field. Let us choose the frame which is locally geodesic at the point $x = 0$. Near this point the metric tensor can be expanded into series, its first two terms being (see, e.g., Ref. [12])

$$g_{ik} = \eta_{ik} + \frac{1}{3} R_{ilmk} x^{l} x^{m} + \ldots \tag{9}$$

where η_{ik} is the metric tensor of the flat space, $R_{iklm} \equiv R_{iklm}(0)$. When calculating the anomaly, not only the terms are inessential that contain the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R, but those as well that contain the derivatives of the Riemann tensor. We can also omit the quadratic term in the metric expansion proportional to $R^{\lambda}_{\alpha\mu\beta\lambda}R_{\gamma\nu\delta}x^{\alpha}x^{\beta}x^{\gamma}x^{\delta}$, since at the following contraction of the Riemann tensors with $\varepsilon^{\mu\nu\alpha\beta}$ it will certainly vanish. With these considerations taken into account equ. (8) for the Green's function, to be more exact, for that its part which contributes to the anomaly, reads as

$$\left(\frac{\partial^2}{\partial x^i \partial x_i} + M^2\right) D_{\nu\rho}(x, y) +$$

$$+ 2\frac{\partial}{\partial x_\alpha}\left[\left(h_{\alpha\beta}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\sigma}_{\alpha\nu}\right) D_{\sigma\rho}(x, y)\right] = i\eta_{\nu\rho}\delta^4(x - y) . \quad (10)$$

Here $h_{\alpha\beta} = \frac{1}{3} R_{\alpha\lambda\delta\beta}x^{\lambda}x^{\delta}$, $\Gamma_{\sigma,\alpha\nu} = -\frac{1}{3}(R_{\sigma\alpha\nu\lambda} + R_{\sigma\nu\alpha\lambda})x^{\lambda}$, all the contractions are made by means of $\eta_{\alpha\beta}$.

We shall go over to the momentum representation ($x^{\mu} \to -i\frac{\partial}{\partial p^{\mu}}$, $\frac{\partial}{\partial x^{\mu}} \to -ip_{\mu}$) and solve Eq. (10) by iterations in the Riemann tensor up to the second order included:

$$D_{\sigma\rho}(p, y) = D^0_{\sigma\rho}(p, y) + D^1_{\sigma\rho}(p, y) + D^2_{\sigma\rho}(p, y) + \ldots$$

$$D^0_{\sigma\rho}(P, y) = -i\frac{\eta_{\sigma\rho}}{p^2 - M^2}e^{ipy},$$

$$D^1_{\sigma\rho}(p, y) = -\frac{2}{3}\frac{1}{p^2 - M^2}p_{\alpha}(R_{\delta\alpha\sigma\lambda} + R_{\delta\sigma\alpha\lambda})\frac{\partial}{\partial p_\lambda}D^0_{\delta\rho}(p, y),$$

$$D^2_{\sigma\rho}(p, y) =$$

$$= -\frac{2}{3}\frac{1}{p^2 - M^2}p_{\alpha}(R_{\delta\alpha\sigma\lambda} + R_{\delta\sigma\alpha\lambda})\frac{\partial}{\partial p_\lambda}D^1_{\delta\rho}(p, y) - \quad (11)$$

$$-\frac{2}{9}\frac{1}{p^2 - M^2}p_{\beta}R_{\alpha\lambda\gamma\beta}(R_{\delta\alpha\sigma\nu} + R_{\delta\sigma\alpha\nu})\frac{\partial^3}{\partial p_\lambda \partial p_\gamma \partial p_\nu}D^0_{\delta\rho}(p, y) .$$

Now we substitute $D^2_{\sigma\rho}(p, y)$ found in this way into Eq. (7), differentiate in x^{μ} and y^{α} (that is multiply by $-ip_{\mu}$ and ip_{α} respectively) and put finally $y=0$. Then, anticipating the integration over d^4p, we average over the directions of the momentum p. As a result, we get

$$\langle\nabla_{\mu}K^{\mu}_{reg}(p)\rangle = -\frac{i}{2}M^2\frac{p^2}{(p^2 - M^2)^4}R_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta} . \quad (12)$$

We come back now to the coordinate representation and take off the regularization by means of the limiting transition $M \to \infty$.

Finally,

$$\langle\nabla_{\mu}K^{\mu}(0)\rangle = \lim_{M\to\infty}\langle\nabla_{\mu}K^{\mu}_{reg}(0)\rangle =$$

$$= -\frac{i}{2}\lim_{M\to\infty}M^2\int\frac{p^2}{(p^2 - M^2)^4}\frac{d^4p}{(2\pi)^4}R_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta} = -\frac{1}{96\pi^2}R\tilde{R} , \quad (13)$$

in complete agreement with expression (1) for the anomaly.

3. We present now shortly the analogous derivation of the chiral anomaly for a spin 1/2 particle in an external

gravitational field. In the momentum representation the contribution of the second order in the external field to the Green's function of the Dirac equation constitutes

$$S_F(p,\,0) = -\frac{\hat{p} + m}{(p^2 - m^2)^5}\left[\frac{R^2}{144}(-3p^2\,\hat{p}m - 3m^3\hat{p} - p^2m^2 - m^4) + \right.$$

$$+ \frac{R_{\mu\nu}R^{\mu\nu}}{360}\,m^2(5p^2 - 45m\hat{p} - 16m^2) +$$

$$\left. + \frac{R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}}{240}\,m^3(-m - 15\hat{p}) + \frac{iR\tilde{R}}{16}\,m^3(\hat{p} - m)\,\gamma^5\right]. \quad (14)$$

Here $\hat{p} = p_\alpha\gamma_\alpha$, γ_α are the Dirac matrices in the flat space, all the contractions are performed by means of $\eta_{\mu\nu}$.

We regularize the chiral current

$$a_\mu = \bar{\psi}\,\gamma^\mu\gamma^5\psi \quad (15)$$

according to Pauli-Villars:

$$a_\mu^{reg}(x) = a_\mu(x) - a_\mu^M(x)\,, \quad (16)$$

where a_μ^M is the chiral current of the regulator particle with the mass M. Due to the motion equations, the vacuum expectation value for the divergence of the regularized chiral current in an external gravitational field is

$$\langle\partial_\mu a^\mu_{reg}(x)\rangle = -\langle\partial_\mu a^\mu_M(x)\rangle =$$

$$= -2Mi\;Sp\,(\gamma^5 S_F(x))\,. \quad (17)$$

Substituting into this formula expression (14) and taking off the regularization by means of the limiting transition $M \to \infty$, we come to the following result for the fermion chiral anomaly:

$$\langle\partial_\mu a^\mu(0)\rangle = \lim_{M\to\infty} -\langle\partial_\mu a^\mu_M(0)\rangle =$$

$$= \lim_{M\to\infty} -2Mi\;Sp\,[\gamma^5 S_F(0)] = -\frac{1}{192\pi^2}\,R_{\mu\nu\alpha\beta}\,\tilde{R}^{\mu\nu\alpha\beta}\,. \quad (18)$$

It is in agreement with the known results of Refs [13 - 15].

The presented parallel derivation by means of the ultraviolet regularization of the chiral anomaly for the spins 1/2 and 1, together with the parallel consideration of these effects at the infrared regularization [1, 10], elucidates the close nature of both anomalies.

REFERENCES

1. *A.D. Dolgov, V.I. Zakharov, I.B. Khriplovich.* Pis'ma v ZhETF, 45 (1987) 511 (Soviet Physics JETP Letters).
2. *R. Endo, M. Takao.* Progr. Theor. Phys., 78 (1987) 440.
3. *A.D. Dolgov, V.I. Zakharov.* Nucl.Phys., B27 (1971) 525.
4. R.E. Kallosh. Pis'ma v ZhETF, 37 (1983) 509 (Soviet Physics JETP Letters).
5. *A.D. Dolgov, I.B. Khriplovich, A.I. Vainshtein, V.I. Zakharov.* Nucl. Phys., B313 (1989) 73.
6. *M.A. Shifman, A.I. Vainshtein.* Nucl. Phys., B227 (1986) 456.

7. *A.A. Ansel'm, A.A. Ioganson.* ZhETF, 96 (1989) 1181.
8. *M.A. Shifman, A.I. Vainshtein.* Preprint, Bern University, June 1990.
9. *M. Reuter.* Phys. Rev., D37 (1987) 1456.
10. *A.D. Dolgov, I.B. Khriplovich, A.I. Vainshtein, V.I. Zakharov.* Nucl. Phys., B315 (1988) 138.
11. *S. Deser, C. Teitelboim.* Phys. Rev., D13 (1976) 1592.
12. *A.S. Eddington.* The Mathematical Theory of Relativity. (Cambridge University Press, 1924.)
13. *T. Kimura.* Progr. Theor. Phys., 42 (1969) 1191.
14. *R. Delbourgo, A. Salam.* Phys.Lett., B40 (1972) 381.
15. *T. Eguchi, P.O. Freund.* Phys. Rev. Lett., 37 (1976) 1251.

A.D. Dolgov, S.V. Kostyuk, I.B. Khriplovich

On the Photonic Chiral Anomaly in an External Gravitational Field

А.Д. Долгов, С.В. Костюк, И.Б. Хриплович

К вопросу о фотонной киральной аномалии во внешнем гравитационном поле

Ответственный за выпуск: С.Г. Попов

Работа поступила 13 марта 1991 г.
Подписано к печати 8.04 1991 г.
Формат бумаги 60×90 1/16
Объем 0,9 п.л., 0,8 уч.-изд.л.
Тираж 220 экз. Бесплатно. Заказ N 31.

Ротапринт ИЯФ СО АН СССР,
г. Новосибирск, 90.