7

ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

A.R. Zhitnitsky

TORONS, CHIRAL SYMMETRY BREAKING AND U(1) PROBLEM IN σ-MODEL AND GAUGE THEORIES.
II. THE GAUGE THEORIES

PREPRINT 89-148

НОВОСИБИРСК

Institute of Nuclear Physics

A.R.Zhitnitsky

TORONS, CHIRAL SYMMETRY BREAKING AND U(1) PROBLEM IN σ-MODEL AND GAUGE THEORIES
II. THE GAUGE THEORIES

PREPRINT 89-148

NOVOSIBIRSK
1989

5. Torons in SU(2) supersymmetric gluodynamics.

We pass now to the analysis of the gauge theories and to corresponding toron solutions. For this purpose the self-dual equation for the gluodynamics will be formulated on the language analogous to Cauchy-Riemann condition for O(3) σ model. The corresponding criteria, which are discussed above for O(3) σ model in terms with local gauge invariance, will help us to choose the "correct" zero modes and to calculate the toron mesure and chiral condensate (see next section) in the supersymmetric gluodynamics case.

We begin with the formulation of the self-dual equation in the Witten's Ansatz [31]:

$$A_0^a = n^a A_0(r,t)$$

$$A_i^a = \epsilon_{iak} n_k \cdot \frac{1+\varphi_2}{r} + (\delta^{ai} - n^a n^i)\frac{\varphi_1}{r} + n^a n^i A_1 \qquad (50)$$

$$A_\mu = (A_0, A_1), \quad n_a = x_a/r, \quad r = (x_i x_i)^{1/2}; \quad \mu = 0,1; \; i = 1,2,3$$

Here i, j, k refer to the three spatial dimensions, and "a" is the isospin index. The precise definitions of $\varphi_{1,2}$ and $A_{0,1}$ are chosen for future convenience. These functions depend only on r and t. Given (50), one readily calculates the field tensor $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - \epsilon^{abc} A_\mu^b A_\nu^c$. The self-dual equations $G_{\mu\nu} = \tilde{G}_{\mu\nu}$ or, equivalently $G_{0i}^a = \frac{1}{2}\epsilon_{ijk} G_{jk}^a$ now become

$$D_\mu \varphi = i\epsilon_{\mu\nu} D_\nu \varphi, \quad D_\mu \varphi = (\partial_\mu + iA_\mu)\varphi, \quad \varphi = \varphi_1 - i\varphi_2, \qquad (51)$$

$$\partial_\mu \partial_\mu \ln \frac{|\varphi|}{r} = \frac{|\varphi|^2}{r^2}, \quad \mu = 0,1$$

Here ∂_0 denotes $\partial/\partial t$ and ∂_1 denotes $\partial/\partial r$. We see that ansatz (50) leads to a two-dimensional Abelian Higgs system. The form of (51) suggests that I regard φ as a charged scalar interacting with the two-dimensional Abeilian gauge field A_μ, with covariant derivative $D_\mu \varphi$.

In this case it is more convenient to study the solutions of the eg.(51) by considering the complex plane and defining:

$$z = r + it, \quad \partial = \partial/\partial z = \tfrac{1}{2}(\partial_1 - i\partial_0), \quad \varphi = \varphi_1 - i\varphi_2$$

$$\bar{z} = r - it, \quad \bar{\partial} = \partial/\partial\bar{z} = \tfrac{1}{2}(\partial_1 + i\partial_0), \quad A = A_1 - iA_0 \qquad (52)$$

It can be easily verified that the most general solution of (51) is [31]:

$$\Psi = \ln \frac{z+\bar{z}}{1-\bar{g}g}, \quad f = \frac{dg}{dz}, \quad g = g(z), \quad \varphi = f e^{\Psi} \qquad (53)$$

$$A = A_1 - iA_0 = -2i\partial\Psi = -2i\partial \ln \bar{\varphi}$$

where $g(z)$ is an analytic function. For Ψ to be nonsigular, we require:

$$|g| = 1 \text{ for } r = 0, \quad |g| < 1 \text{ for } r > 0 \qquad (54)$$

Thus the expressions (53) for $\varphi = \varphi_1 - i\varphi_2$ and for $A = A_1 - iA_0$ with extra requirements (54) solve our problem. Namely, the self-dual solution is formulated in terms of analytical function $g(z)$ analogous to the O(3) σ model (3). The extra requirement (54) retlects the fact, that our theory is defined only on the half plane $z + \bar{z} \geq 0$, while the O(3) σ model is defined on the full complex plane, of course.

To determine the solution with topological charge Q= 1/2, we have to express the four-dimensional topological charge in terms 2d fields [31]:

$$Q = \frac{1}{32\pi^2}\int d^4x G\tilde{G} = \frac{1}{2\pi}\int_0^\infty dr \int_{-\infty}^{\infty} dt \left[\tfrac{1}{2}\epsilon_{\mu\nu} F_{\mu\nu} + i\epsilon_{\mu\nu}\partial_\mu(\bar{\varphi} D_\nu \varphi)\right] = \qquad (55)$$

$$= \frac{1}{2\pi i}\oint ds \frac{d}{ds}\ln f; \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

It is easy to see from (55) that the topological charge is determined by the change of phase function $f(z)$ around the contour which encloses the region ReZ $\geq$ 0.

Now all is ready for the description of the self-dual solutions in complete analogy with the discussed above consideration in the O(3) σ model (sections2,3).

The solution $g(z) = (a-z)/(\bar{a}+z)$ satisfy to the finiteness condition (54) for any complex number a with Rea > 0. In this case an easy calculation shows that this solution is a gauge transform of the vacuum; the solution

$$g(z) = \prod_{i=1}^{2}\left(\frac{a_i - z}{\bar{a}_i + z}\right) \qquad \mathrm{Re}\, a_i > 0$$

describes the instanton [2] (but in different gauge) with Q=1. The toron solution with Q = 1/2 (by anology with O(3) σ model (11)) is described by the function

$$g(z) = \lim_{(a\cdot\bar{a})\to 0}\left(\frac{a-z}{\bar{a}+z}\right)^{3/2}, \quad a = \Delta + it_0, \quad \mathrm{Re}\, a = \Delta > 0 \qquad (56)$$

The solution $g(z)$ (56) turns out in a sense between the vacuum (Q=0) and instanton (Q=1) solutions described above. Here $\Delta = f(a+\bar{a}) \to 0$ is a regulator, analogous to the dimensional parameter Δ (13) in O(3) σ model. This solution like to σ model, is defined on two Riemann sheets; real physical space corresponds to but one of them and can be understood as manifold with boundary. If one sets $\Delta = 0$ from the very beginning, then $g(z) = 1$ corresponding to the empty vacuum solution.

However, the solution (56) is nontrivial. As will be shown in the next section, the solution (56) ensures nonzero value of the gluino condensate in SYM. Analogous behaviour, as was discussed above, arises in SUSY O(3) σ model.

Obviously, the solution (56) satisfy to the finituess condition (54) for $\Delta > 0$. The imaginary part of the parameter "a" ($\mathrm{Im}\, a = t_0$) determines the location of the toron along the time axis, and without loss of generality we take in what follows $\Delta = 1$, $t_0 = 0$. Moreover, upon completion of the large circle in the physical space (Re $z \geqslant 0$) the function $f(z)$ acquires a phase π, which according to (55) corresponds to Q = 1/2. In the next section we calculate the value Q by using gauge-invariant approuch. Like in σ - model, the limit of the solution (56) with a cut, tending to zero as $\Delta \to 0$ means reestablishment of single-valuedness on one physical sheet for the gauge field A_μ^a. The gauge - invariant values (like $G_{\mu\nu}^2$) are single-valued at any finite Δ (not only in limit $\Delta \to 0$) So, the solution (56) can be understood as a point defect with the Δ - reqularization which is preserved the duality equation at finite value Δ.

Thus, the Eq.(53) with analytic function $g(z)$(56) provides the solution of $G_{\mu\nu} = \tilde{G}_{\mu\nu}$ with finite action and with fractional topological charge. However, the explicit expressions (53) for A, φ are not so usefull for the future calculations because the complexity their forms. So, we would like to find the appropriate gauge transformations and to make change the variable for the next analysis.

The rest of the Section is a series of technical comments, concerning some implacations of eq.(53), (56). First, we consider the explicit expression for $G_{\mu\nu}^2$. Substituting (50-

53) in $G_{\mu\nu}^2$ we obtain:

$$G^2 = G\tilde{G} = \frac{4}{r^2}|D_\mu \varphi|^2 + F_{\mu\nu}^2 + \frac{2}{r^4}\left(1-|\varphi|^2\right)^2 \qquad (57)$$
$$|D_\mu \varphi|^2 = 2|\varphi|^2 \partial \ln|\varphi|^2 \bar{\partial} \ln|\varphi|^2 = \frac{1-|\varphi|^2}{r^2}|\varphi|^2 + 2\partial\bar{\partial}|\varphi|^2$$
$$F_{\mu\nu}^2 = 2(-4\partial\bar{\partial}\psi)^2 = 2\left(\frac{1-|\varphi|^2}{r^2}\right)^2$$

It is easily to see that the $G_{\mu\nu}^2$ depends only on the absolute value of the field φ. So we make the following change of variables:

$$z = -i\,\mathrm{ctg}\,\omega, \quad \omega = u + iv, \quad \partial = -\frac{i}{1-z^2}\partial_\omega = -i\sin^2\omega\,\partial_\omega \qquad (58)$$

Here (- V) (0 $\leqslant$ (-V) $\leqslant \infty$) plays role of the measure of length and (U) ($-\pi/2 \leqslant u \leqslant \pi/2$) is the angle-variable, see fig.5. In terms of the variables U, V, the absolute value of $|\varphi|$ depends on " v " only and does not depend on angle-variable "U" (just it is the reason of the change of variables (58)):

$$|\varphi| = \frac{3}{2}\frac{\mathrm{sh}\,2v}{\mathrm{sh}\,3v}, \quad |\varphi|_{v\to 0} = 1 - \frac{5}{6}v^2 \to 1, \quad |\varphi|_{v\to -\infty} = \frac{3}{2}e^{v} \to 0 \qquad (59)$$

We are now ready to describe the dependence of $G_{\mu\nu}^2$ on r,t. at $r \to 0$ and $|z| \to \infty$. In this case the parameter v tends to zero:

$$\mathrm{th}\,2v \simeq 2v = -\frac{2r}{1+|z|^2}, \quad v \to 0 \qquad (60)$$

Substituting (58-60) to the expression for $G_{\mu\nu}^2$ (57) we obtain:

$$G_{\mu\nu}^2(r\to 0) \sim \frac{1}{(1+t^2)^4}, \quad G_{\mu\nu}^2(z\to\infty) \sim 1/|z|^8 \qquad (6_1)$$

The solution (56) possess finite action, but in our gauge the four-dimensional gauge field A_μ^a is actually singlural at r = 0. It is necessary to perform a gauge transformation on the solution to satisfy regularity coudition; such a transformation always exists because $|\varphi|^2 = 1$ at $r = 0$, see Eq.(59). We shall discuss the corresponding gauge below. Now we shall cousider behaviour of $G_{\mu\nu}^2$ on the cut. It easily to see that the gauge-invariant value $G_{\mu\nu}^2$ on the upper edge of the cut coinsides with the value on the lower edge and has an integrable sin ularity at z tends to 1 ($v \to -\infty$):

$$G_{\mu\nu}^2(z\to 1) \sim \frac{|\varphi|^2}{(1-z^2)(1-\bar{z}^2)} \sim \frac{e^{2v}}{|1-z|^2} \sim \frac{1}{\sqrt{(1-z)(1-\bar{z})}} \qquad (62)$$

The origin of this physical singularity is connected with fractional topological number of our solution. This singularity plays an important role because it will help us choose the "correct" zero modes in the toron background.

Using (57), (59) we can calculate the action and topological charge in a more direct, gauge-invariant way:

$$Q = \frac{Sg^2}{8\pi^2} = \frac{1}{32\pi^2}\int d^4x\, G_{\mu\nu}^2 = \frac{1}{32\pi^2}\int 4\pi^2 r^2 dr\,dt\Big[\frac{4}{r^2}\,\frac{1-|\varphi|^2}{r^2}|\varphi|^2 + \tag{63}$$
$$+\frac{8}{r^2}\partial\bar\partial|\varphi|^2 + \frac{4}{r^4}(1-|\varphi|^2)^2\Big] = \frac{1}{2\pi}\int dr\,dt\left(\frac{1-|\varphi|^2}{r^2} + 2\partial\bar\partial|\varphi|^2\right).$$

The secoud term is a total divergence, and so can be written as a boundary integral which vanishes because for solution (56) $\partial_\mu|\varphi| = 0$ at the boundary of the space volume at $v \to 0$ (59). From expression (59) for $|\varphi|$ I find that the first term (63) becomes:

$$Q = \frac{Sg^2}{8\pi^2} = \frac{1}{2\pi}\int dr\,dt\,\frac{1-|\varphi|^2}{r^2} = \frac{4}{2\pi}\int_{-\pi/2}^{\pi/2} du\int_{-\infty}^{0} dv\,\frac{1-|\varphi|^2}{sh^2 2v} = \tag{64}$$
$$= \frac{1}{2}\int dv\left[\frac{4}{sh^2 2v} - \frac{9}{sh^2 3v}\right] = \frac{1}{2}$$

in agreement with result which was described above.

Now we would like to find the gauge transformation which allows to write down the solution (56) in the superpothential form:

$$A_\mu^a = -\bar\zeta^a_{\mu\nu}\,\partial_\nu \ln P(r,t) \qquad \epsilon_{0123} = 1 \tag{65}$$
$$\bar\zeta^a_{0i} = \delta^a_i,\quad \bar\zeta^a_{i0} = -\delta^a_i,\quad \bar\zeta^a_{ij} = -\epsilon^{aij},\quad \bar\zeta^a_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\bar\zeta^a_{\lambda\sigma}$$

Here $\bar\zeta^a_{\mu\nu}$ are the 't Hooft symbols [32]*, and P is superpotential, depending only on r and t. For this let us remined that the duality equation (51) possesses a remaining gauge invariance with any analytic function h(z) [31]:

$$f'(z) = f(z)h(z),\quad \varphi' = \varphi\, h(z)/|h| = \varphi\exp(i\alpha) \tag{66}$$
$$\psi' = \psi - \frac{1}{2}\ln\bar h h,\quad A' = A + i\,\partial\ln|h|^2$$

Using complex variables the Eq.(65) can be rewritten

$$\frac{\varphi' - i}{r} = A' = 2i\,\partial\ln P(r,t) \tag{67}$$

*The difference with the standard definition [32] is connected with nonstandard expression for $G_{\mu\nu}$ (50).

It can be easily verified that (65) and (66) are solved by

$$h(z) = [i+g(z)]^{-2},\quad P^{-1} = \frac{z+\bar z}{1-\bar g g}|i+g|^2 = e^{\psi'} = e^{\psi}|i+g|^2 \tag{68}$$

In fact, substituting (68) to the (53), (66), we obtain the following relations:

$$A' = -2i\partial\psi - 2i\partial\ln(i+g) = -2i\partial\psi - 2i\frac{1}{i+g}\partial g =$$
$$= -\frac{2i}{2r} + 2i\left[\frac{-\bar g}{1-\bar g g} - \frac{1}{i+g}\right]\partial g = -\frac{i}{r} + 2\frac{|i+g|^2(\partial g)}{(1-\bar g g)(i+g)^2} = \tag{69}$$
$$= -\frac{i}{r} + (fe^{\psi})\frac{1}{r}\frac{|i+g|^2}{(i+g)^2} = -\frac{i}{r} + \frac{1}{r}\varphi'$$

It is easy to see that relation (69) is precisely superpotential solution (65), (67). Moreover, as will be shown below, the solution in superpotential form (65) is regular at r = 0; because φ' (r = 0) = i, and hence φ_1' (r = 0) = 0, φ_2' (r = 0) = -1 and A_μ^a (50) is regular at r = 0. In the following relations the superpotential solution (65) will be used only and corresponding primes will be removed.

It is instructive to check the result (65) with superpotential P (68) in a more direct way. As well-known the self-duality equation for the gauge (65) can be written as follows [33]

$$P^{-1}\Box P = 0 \tag{70}$$

Here $\Box$ is the four-dimensional Laplacian. By algebraic manipulations this can be proved, using the relations (57) and (66):

$$P^{-1}\Box P = P^{-1}(\partial_0^2 + \partial_1^2 + \frac{2}{r}\partial_1)P = 4P^{-1}\partial\bar\partial P + 2/r\, P^{-1}(\partial+\bar\partial)P =$$
$$= 4(\partial\bar\partial\ln P + \partial\ln P\,\bar\partial\ln P) + \frac{2}{r}\partial\ln P + \frac{2}{r}\bar\partial\ln P =$$
$$= \frac{1}{r^2}(1-|\varphi|^2) + \frac{(\varphi-i)(\bar\varphi+i)}{r^2} + \frac{\varphi-i}{ir^2} - \frac{\bar\varphi+i}{ir^2} = 0 \tag{71}$$

In conclusion of this section I would like to analyse the superpotential P and field φ in the gauge (65). Let us consider the P, φ functions in ω-variable. From (58), (66), (68) we have:

$$P^{-1} = \frac{2\sin(\omega-\bar\omega)}{\sin\frac{3}{2}(\omega-\bar\omega)}\cdot\frac{\sin\frac{3}{2}\omega\cdot\sin\frac{3}{2}\bar\omega}{\sin\omega\,\sin\bar\omega} \tag{72}$$
$$\varphi = \frac{3}{2}i\,\frac{\sin(\omega-\bar\omega)}{\sin\frac{3}{2}(\omega-\bar\omega)}\,\frac{\sin\omega}{\sin\bar\omega}\,\frac{\sin\frac{3}{2}\bar\omega}{\sin\frac{3}{2}\omega}$$

Let us summarize some important properties of these functions. The discussion of these properties will facilitate us in the next section the choice of the "correct" zero modes in the toron background. First, we note that the P-function is regular, single-valued function, tending to constant as $|z|\to\infty$. Then, as is easily verified from the explicit expression (72), the $\mathcal{P}$ -function tends to (i) when γ tends to zero. As mentioned above the gauge potential A_μ^a is therefore reqular at $z = 0$. Besides that, the imaginary part of $\mathcal{P}$ ($Im\,\mathcal{P} = -\mathcal{P}_2$) is single-valued function and real part ($Re\,\mathcal{P} = \mathcal{P}_1$) has a different sign on the different edges of the cut. Thus the expression

$$\bar{A}A = \left(\frac{\mathcal{P}-i}{z}\right)\left(\frac{\bar{\mathcal{P}}+i}{\bar{z}}\right) = \frac{1}{r^2}\left[\mathcal{P}_1^2 + (1+\mathcal{P}_2)^2\right] \tag{73}$$

is regular and single-valued function.

The proporties of the discussed above functions: $G_{\mu\nu}^2, P, \mathcal{P}$ are very important for further analysis.

6. Zero modes, toron mesure and gluino condensate in SYM.

As in known, supersymmetric models differs conveniently from ordinary ones in that only zero modes need be considered. This because the nonzero modes cancel between bosous and fermions, and the mesure is defined by the zero modes only.

Usually the explicit determination of zero modes in any self-dual field is a simple exercise because they can be expressed in terms of the field strength tensor $G_{\mu\nu}^a$. Indeed, in the $D_\mu^{cl.} a_\mu$ - gauge zero modes satisfy to the following equation [32]:

$$\left[(-D^2)^{ac} g_{\mu\nu} + 2\epsilon^{abc} G_{\mu\nu}^b\right] a_\nu^c = 0 \quad , \quad D_\mu^{ac} a_\mu^c = 0 \tag{74}$$

$$D_\mu^{ac} = \delta^{ac}\partial_\mu - \epsilon^{abc} A_\mu^b \, , \quad [D_\mu D_\nu]^{ac} = -\epsilon^{abc} G_{\mu\nu}^b$$

Here A_μ^a is the classical field and a_μ^a is the small quantum fluctuation, μ - is the four - dimensional index. It is easy to prove that 4 translation modes, 3 gauge modes and 1 conformal mode can be expressed in terms $G_{\mu\nu}^a$ and they satisfy to Eq.(74):

$$a_\mu^a(\lambda) \sim G_{\mu\lambda}^a \qquad \lambda = 0,1,2,3$$

$$a_\mu^a(d) \sim G_{\mu\nu}^a \bar{\eta}_{\nu\lambda}^d x_\lambda \quad , \quad d = 1,2,3$$

$$a_\mu^a \sim G_{\mu\nu}^a x_\nu \tag{75}$$

In particular, in instanton field $G_{\mu\nu}^a \sim \eta_{\mu\nu}^a \frac{1}{(1+x^2)^2}$, we obtain well-known 8 zero modes [32]

$$a_\mu^a(\lambda) \sim \eta_{\mu\lambda}^a \frac{1}{(1+x^2)^2}, \quad a_\mu^a(d) \sim \eta_{\mu\nu}^a \bar{\eta}_{\nu\lambda}^d \frac{x_\lambda}{(1+x^2)^2}, \quad a_\mu^a \sim \eta_{\mu\nu}^a \frac{x_\nu}{(1+x^2)^2} \tag{76}$$

These modes are normalizable and reqular and thus they satisfy to all requirements which was discussed above (see Sect.3).

In the toron case under consideration with topological charge equal one-half, zero modes (75) still satisfy to the Eq.(74). However, they do not satisfy to the regularity condition and so they are unadmissible modes. In fact, because $G_{\mu\nu}^2$ (62) has an sinqularity at $\bar{z} \to 1$, then the same singularity have a zero modes (75). Thus, they are forbidden in the toron case.

This situation was expected beforehand from the corresponding analysis of the O(3) σ model. Let us remind that the zero modes (75) are derivatives of the classical solution up to a gauge transformation. In instanton case it is a correct way to obtain zero modes. But in toron case it is not so, that can be seen from consideration of zero modes in O(3) σ model. In this case the toron solution look as follows $\varphi_{cl.} \sim \sqrt{1/z-a}$ (13). The "natural" zero mode which is the derivative of $\varphi_{cl.}$ with respect to collective coordinate "a" is $\delta\varphi_0 \sim \partial/\partial a\, \varphi_{cl} \sim (z-a)^{-3/2}$. Such function satisfies to the corresponding equation for zero modes $\bar{\partial}(\delta\varphi_0) = 0$ (28). However this function does not satisfies to the regularity condition (29) and hence it does not acceptable. As known the correct zero mode is $\delta\varphi_0 \sim z^{-1}$ (34).

The lesson from this is follows. The "natural" zero modes satisfy to corresponding equations. But they are forbidden because of the regularity requirement. To find the correct zero modes, the necessity to solve the corresponding equation (74) by explicit way and to shoosé the correct zero modes, satisfying the regularity requirement.

But, as is well known [34], the vector field equations for the small fluctuations about a self-dual field are equivalent to the Dirac equation for a spinor with unit isospin and with definite chirality. The problem of counting the number of the modes of massless excitations of the vector field is thus reduced to that of counting the number of massless modes of the Dirac field. But, the number of the fermion zero modes is known beforehand, from consideration of the axial anomaly. So, for each solution of the spinor field equation there are precisely two linearly independent solutions of the vector field equation [34].

For instance, for the instanton we expect four fermion zero modes in accordance with the fact that the solution with Q = 1 changes the chiral charge ΔQ_5 by four units and can ensure a nonvanishing value for the correlator $\langle \bar{\psi}\psi(x)$, $\bar{\psi}\psi(0)\rangle$ [7, 8]:

$$\psi^a_{1,2} \sim \sigma^+_\mu G^a_{\mu 0}\,\varepsilon \quad , \quad \psi^a_{3,4} \sim \sigma^+_\mu G^a_{\mu\nu} x_\nu \varepsilon \tag{77}$$

So, from [34] we expect 8 real vector zero modes in agreement with explicit calculation (75).

For toron we expect two (real) fermion zero modes with unit isospin and so we expect four (real) vector zero modes in agreemeut with four collective coordinates associated with translations of toron solution. We remind that the parameter Δ is the regulator, but it does not has a sense of collective coordinate. Because the toron solution with Q = 1/2 changes the chiral charge by two units and has (as will be explicitey shown bellow) two gluino zero modes, the corresponding vacuum transition is necessarily accompanied by the production of a $\bar{\psi}\psi$ pair. So we expect that toron can ensure a nonvanishing value for the condensate $\langle\bar{\psi}\psi\rangle$.

We return to the analysis of Eq.(74). As was discussed above, these equations can be rewritten in terms spinor field with isospin one. To make this construction explicit, we note that if

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix}$$

is a solution of the spinor field equation:

$$D_\mu \sigma^-_\mu \psi = 0\ , \quad \sigma^\pm_\mu = (\pm i, \vec{\sigma})\ , \quad \sigma^-_\mu \sigma^+_\nu = g_{\mu\nu} - i\bar{\xi}^a_{\mu\nu}\sigma^a \tag{78}$$

then two real vector modes:

$$a_\nu = -\frac{1}{2}\,\mathrm{Tr}(\sigma^+_\nu \Phi)\ , \quad \Phi^{(1)} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}, \quad \Phi^{(2)} = \begin{pmatrix} ia & ib^* \\ ib & -ia^* \end{pmatrix} \tag{79}$$

satisfy to starting equation (74), see Ref.[34].

Now we are ready to find the explicit solution of Dirac equation (78). For this we consider the toron solution in the gauge (65) $A^a_\mu = -\bar{\xi}^a_{\mu\nu}\,\partial_\nu \ln P(r,t)$, and look for the solution of (78) in the corresponding form:

$$\psi^c \sim \bar{\xi}^c_{\lambda\mu}\,\sigma^+_\mu\,\partial_\lambda P \cdot f(P)\cdot\varepsilon \tag{80}$$

Here ε is the constant Lorentz spinor, P is the superpotential (68) and lastly $f(P)$ is some function which will be found now.

Substituting the expression (80) to the Dirac Eq.(78) we obtain the following equation:

$$\left[\partial_\mu \delta^{ac} + \epsilon^{abc}\bar{\xi}^b_{\mu\nu}\,\partial_\nu \ln P\right]\sigma^-_\mu \sigma^+_\lambda \bar{\xi}^c_{\lambda\sigma}\cdot\partial_\sigma P\cdot f(P)\cdot\varepsilon = 0 \tag{81}$$

Using the proporties of $\sigma^\pm_\mu$ -matrix (78) and $\bar{\xi}^a_{\mu\nu}$ -symbols [32] the Eq.(81) can be rewritten in the following form:

$$(\Box P)\cdot f + (\partial_\mu P)^2\left[\frac{\partial f}{\partial P} + \frac{2f}{P}\right] = 0$$

Because P-function satisfies to Eq.(70), the f-function is equals:

$$f(P) = P^{-2}$$

Substituting this solution in (80) we obtain the following expression for two gluino zero modes:

$$\psi^a \sim \bar{\xi}^a_{\lambda\sigma}\,\sigma^+_\lambda(\partial_\sigma \ln P)P^{-1}\varepsilon \sim \sigma^+_\mu A^a_\mu P^{-1}\cdot\varepsilon\ , \quad \varepsilon = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \tag{82}$$
$$|\psi^a|^2 \sim P^{-2}A^a_\mu A^a_\mu = 3P^{-2}\bar{A}A = 3P^{-2}\left|\frac{\Phi - i}{z}\right|^2$$

In obtaining (82) we took into account the expression for toron solution (65) and the expression for potential in complex notation (52). Now we are ready to prove that the solution (82) is regular, singlevalued and square-integrable zero mode. For this purpose let us remind that P is the regular function tending to constant for $|z| \to \infty$. Then $\left|\frac{\Phi - i}{z}\right|^2$ is the regular function at critical points $z = 0$, $z = 1$. Moreover, the gauge-invarint value $|\psi^a|^2$ is a singlevalued function because property (73), lastly, as can be shown from Eq.(72), the $|\psi|^2$ tends to zero like $|z|^{-6}$ at $|z| \to \infty$ and so $|\psi|^2$ is square-integrable. Thus the both of two fermion zero modes satisfy to all requirements discussed above and must be taken into account. In the following relations the normalization

$$\int |\psi|^2 d^4x = 1 \tag{83}$$

is assumed, although not explicitly indicated.

Now we would like to note, that the instanton solution can be expressed in the form (65) as well(so-called singular gauge). In this case superpotential "P" and gauge field A_μ^a are equal respectively:

$$P = \frac{x^2+1}{x^2}, \quad A_\mu^a = 2\bar{\xi}^a_{\mu\nu}\frac{x_\nu}{x^2}\frac{1}{1+x^2}$$

The zero modes (82) now are:

$$\psi^a \sim \sigma_\mu^+ \bar{\xi}^a_{\mu\sigma} \frac{x\sigma}{(1+x^2)^2} \cdot \varepsilon$$

and coinside with expression (77). In toron case the solution (82) does not coinside with the expression (77) and only the solution (82) satisfies all our requirements.

As was mentioned above, the number of the "correct" fermion zero modes is known beforehand, from a consideration of the axial anomaly. Rather, we wanted to demonstrate just the realization of this general index theorem. The explicit calculation (82) confirms our general consideration. Substituting expression (82) for two gluino zero modes into the Eq.(79) we obtain four gluon zero modes satisfying all requirements. This comes from the fact, that the gauge-invariant values for bosonic and fermionic zero modes coinside, as can be seen from (79):

$$|\psi^c|^2 \sim a_\mu^c a_\mu^c$$

An explicit form of these gluon zero modes a_μ^a is unessential for our purpose, however the number of the admissible vector zero modes is very important question. This number is equal to four in agreement with general discussion and in accordance with the existance of four translation collective coordinates.

With the above consideration taken into account the toron measure in SYM acquires the following form:

$$Z_{toron} = C\frac{M_0^4}{g^4}d^4x_0\frac{d^2\varepsilon}{M_0}\exp\left(-\frac{4\pi^2}{g^2}\right) = C\frac{1}{g^4}\Lambda^3_{1\text{-}loop}\, d^4x_0\, d^2\varepsilon \tag{84}$$

$$\Lambda^3_{1\text{-}loop} \equiv M_0^3\exp\left(-\frac{4\pi}{g^2}\right)$$

Here the factor $g^{-4}M_0^4 d^4x_0 \sim (\sqrt{S_{cl.}}\, M_0\, dx_0)^4$ is due to the four bosonic zero modes mentioned above; d^4x_0 is the corresponding integral over the collective variables; the factor $d^2\varepsilon\, M_0^{-1}$ is connected with two fermionic zero modes (82); last-

ly $\exp(-4\pi^2/g^2)$ is the contribution of the classical toron action (64), and "C" is some calculable constant.

As in the case of the instanton calculation [7,8] the expression (84) for the toron mesure has precisely the renorm-invariant form. It is easy to trace this phenomenon, starting from instanton density [7,8,28]:

$$Z_{inst.} = \frac{M_0^8}{g^8}d^4x_0\frac{d^4\rho}{(2\pi^2)}\frac{4}{\pi^2}(4\pi^2)^4 d^2\varepsilon_1 d^2\varepsilon_2 \exp\left(-\frac{8\pi^2}{g^2}\right) \tag{85}$$

Here the factor $\exp(-8\pi^2/g^2)$ is connected with instanton action, $M_0^8 g^{-8}$ is the regulator contribution, corresponding to the eight bosonic zero modes (76); the factor $(d^2\varepsilon_1/M_0)(d^2\varepsilon_2/M_0)$ is connected with four fermionic zero modes (77). While the action decresed by a factor two, the number of admissible zero modes decreased by the same factor, which ensures the correct renorminvariant behaviour. Let us remind that the analogous situation accurs in the σ-model, see Sect. 4.

Now all is ready for the calculation of the gluino condensate in SYM. Substituting in place of ψ their zero modes (82), we verify that

$$\langle g^2\lambda^2\rangle = 2C\frac{1}{g^2}\Lambda^3_{1\text{-}loop} \tag{86}$$

In the last step we used the value of normalization integral (83) and changed the notation of gluino field from ψ^a to λ^a because in next Section we shall cousider the model with fundamental representation of fields (quarks). Namely for quark fields we reserve the ψ -notation.

The nonvanishing of the transition amplitude (86) means the nonvanishing of the corresponding condensate analogously the σ-model consideration (Sect.4). In the case under consideration like in σ-model, the nonzero value of the gluino condensate indicates the spontaneous breaking of discrete chiral symmetry, which does not take place in any order of perturbation theory. The toron solution with Q = 1/2 changes the chiral charge by two units and has two admissible zero modes (82). Therefore the corresponding vacuum transition is necessarily accompanied by the production of a λ^2 pair, as the explicit calculation of (86) also demonstrated. Like in σ-model the correct physical states are the linear superpositions

(45) of the states with the definite chiral charge, and we do not dwell on this issue.

In conclusion of this section we would like to discuss the possibility of the exctraction of the constant C in Eq.(84) from the instanton formula (85). For this let us consider two torons at position z_1 and z_2. For small toron separation we suppose that $\frac{1}{2}(z_1+z_2)=x_0$ can be interpreted as the position of instanton, and $(z_1-z_2)=\rho\to 0$ as the size of this instanton.

Each of these systems (2 torons or 1 instanton) have the same values of the action topological charge and the numbers of fermionic and bosonic zero modes. Moreover, this interpretation is supported by consideration of the expressions for toron (84) and instanton (85) densities. In fact, one can interpret $d^4x_0\, d^4\rho$ from (85) as the integral over two toron collective coordinates (translations z_1 and z_2) because $d^4x_0 d^4\rho = d^4z_1 d^4z_2$*. Now the instanton mesure (85) can be understood as square of toron mesure (84) and so

$$Z_{inst.} = \frac{1}{2!} Z_{tor}(z_1)\cdot Z_{tor.}(z_2) \tag{87}$$

where combinatorial factor $(2!)^{-1}$ is necessary to avoid double counting. From (84,85,87) we obtain:

$$C = 2^5\pi^2 \tag{88}$$

---

*As will be shown below, this interpretation is still valid in other theories, such as SQCD, QCD and so on. Moreover our conjecture be confirmed by consideration of arbitrary gauge group G. In this case, as well known, the number of bosonic zero modes in instanton field is defined by the qudratic Casimir operator C(G) and equal to 4C(G). In particular for SU(N) gauge group C(SU(N)) = N and 4N instanton zero modes can be interpreted as translations of N torons. We think that in this case the admissible value for toron topological charge Q = 1/N [29], like, in CP^{N-1} theories [30]. Besides that, this conjecture is in agreement with the number of the vacuum states (equal N), in SYM and with 2N gluino instanton zero modes, which are necessarily present in this model. So the instanton mesure can be understood as $Z_{inst} \sim (Z_{toron})^N$. This formula, as can be shown, ensure the correct renorm-invariant dependence of Z_{toron} and nonzero value for $\langle\lambda^2\rangle$ in analogy with the result (84) described above for SU(2) gauge group.

Now we are ready to compare our direct calculation of $\langle g^2\lambda^2\rangle$ (86) with strong coupling instanton computation [8, 28] and with an inderect method [10] which allows one to compute $\langle g^2\lambda^2\rangle$ exactly. From the expression (88) for the constant "C" it can be seen, that our result (86) is in agreement with [10] and disagrees with [8, 28], by a factor $\sqrt{4/5}$. We don't yet understand the origin of this difference because our method, in a sense, is the strong coupling computation analogous to the method of Refs. [8, 28].

In addition, in our approach, we have considered the contribution to $\langle g^2\lambda^2\rangle$ just due to the classical toron solution; the instanton solution can ensure a nonvanishing value only for the certain correlator $\langle\lambda^2, \lambda^2\rangle$. Besides that, we have found 2 vacua in SYM (45) with nonvanishing $\langle\lambda^2\rangle$ in the explicit way, in agreement with the Witten index and with the discrete symmetry breaking in this model. The instanton calculation gives only an average over these vacua.

7. Toron mesure in supersymmetric QCD (SQCD).

This model, besides the SYM part we have discessed before, contains the N_f matter fields in the fundamental representation. We shall follow the notation of [28] and consider the toron mesure in SQCD.

In this case, in comparison with the toron measure in SYM (84) we have additionally the two factors d_b and d_f connected with bosonic and fermionic matter fields and with the corresponding quadratic functional untegration around the toron [32]:

$$Z_{SQCD} = Z_{SYM}\,(d_f)^{N_f}(d_b)^{-N_f} \tag{89}$$

$$d_f = \mathrm{Det}\left[\frac{-i\hat{D}-im}{-i\hat{\partial}-im}\right] \qquad d_b = \mathrm{Det}\left[\frac{-D^2+m^2}{-\partial^2+m^2}\right]$$

Here the regulator contribution is taken into account implicitly. Formal manipulations of d_f allows us to write [35]:

$$m\frac{d}{dm}\ln d_f = \mathrm{Tr}\left[\frac{+m^2}{-\hat{D}^2+m^2} - \frac{m^2}{-\hat{\partial}^2+m^2}\right] = \tag{90}$$

$$\mathrm{Tr}\left[\frac{m^2}{-D^2+m^2} - \frac{m^2}{-\partial^2+m^2}\right] - \mathrm{Tr}\left[\frac{m^2\gamma_5}{-\hat{D}^2+m^2}\right].$$

The symbol Tr denotes a trace over space-time, Dirac and color indices. In obtaining (90) we took into account the identity:

$$-\hat{D}^2\left(\frac{1+\gamma_5}{2}\right) = -D^2\left(\frac{1+\gamma_5}{2}\right),$$

which is valid for any self-dual fields. It is well-known [34], [36] that the last term in (90) is connected with the index of Dirac operator, actually independent of "m" and equals to a topological charge Q of background field. This is established most easily from the formal expression of the last term in (90).

$$-Tr\left(\frac{m^2\gamma_5}{-\hat{D}^2+m^2}\right) \equiv T(m^2) \tag{91}$$

Differentiating $T(m^2)$ with respect m^2 we conclude that $\partial/\partial m^2\, T(m^2) = 0$. Thus, $T(m^2)$ is a constant which can be evaluated by the large - m^2 limit. In this limit - $\hat{D}^2$ can be replaced by P^2, where the momentum P_μ has the differential-operator realization $-i\partial_\mu$. Thus, the calculation in this case is a very simple one and yields $T(m^2) = Q$ [34, 36]

The first term in (90) is connected with nonzero modes and cancels with the contribution of d_B corresponding to bosonic nonzero modes. So, from Eq.(89) we have

$$m\frac{d}{dm}\ln Z_{SQCD} = QN_f, \quad Z_{SQCD} = Z_{SYM}\left(\frac{m}{M_0}\right)^{QN_f} \tag{92}$$

In particular, in the instanton case with Q = 1, the expression (92) corresponds simply to zero mode contribution, i.e. $Z_{SQCD} \sim m^{N_f}$ as it should be. So, the $m^2 \to 0$ limit of $T(m^2)$, at integer value Q, receives contributions only from zero modes, and each normalized zero mode gives an exactly unity contribution to T (91).

In the toron case with fractional topological, charge, formula (92) still correct, but the quark's admissible zero modes are absent. The puzzle now is as follows: the RHS of (91) need not be an integer (in our case T = 1/2), and the LHS of (91) is usually determined by zero modes with the integer contribution to eq.(91). The paradox is solved as follows.

Formula (91) is still correct because the existence of contributions to the LHS which come from the continuum. Undeed, let us call $C(\lambda,\tau)$ the spectral density of the corresponding Dirac's operator [37]. The parameter $\tau \to \infty$ in the $C(\lambda,\tau)$ has a sense of infrared regularization and tends to infinity

in the final answer 37 . In this case the eq.(91) can be rewritten as follows

$$\lim_{\tau\to\infty}\int d\lambda\, C(\lambda,\tau)\frac{m^2}{\lambda^2+m^2} = Q = \frac{1}{2} \tag{93}$$

and at $m \to 0$. We have

$$\lim_{\tau\to\infty} C(\lambda,\tau) \to Q\delta(\lambda) \tag{94}$$

This is exactly the result of Ref. [37], where it was argued that the continuum contribution is zero unless there are bound states or unbound resonances at $\lambda = 0$ in the spectrum of Dirac operator. So, we can conclude that in the presence of topologically nontrivial gauge fields, ($Q = 1/2 \neq 0$ in eq.(93)) the spectrum of Dirac operators must include an unbound resonance at $\lambda = 0$. In either case when the system is placed in a box of radius R_0 the lowest eigenvalue will approach zero faster than R_0^{-1}. The argument given by 't Hooft [32] and formula (92) allows us to conclude that the functional integral tends to zero as $Z \sim m^Q = m^{1/2}$ when $m \to 0$. We will call these modes by guasi-zero modes (QZM), because these modes belong to the continuum.

As will be shown below, the Dirac spectrum with the aforementioned properties (the absence of the gap between zero eigenvalue and continuum) play a very important role in the forming of the chiral condensates. On the other hand, these proporties are exactly the features of the solutions with fractional topological number. Let us recall that in the instanton case with integer Q = 1, the LHS of (91) is determined by zero mode and the continuum is separated by the gap.

From a mathematical point of view, this phenomenon is connected with the definition of our solution on the manifold with boundary and the index theorem should be modified (for more detail see Refs. [26, 27] . The equations (91), (93) can be understood from another point of view, namely, from an extension of Levinson's theorem of potential scattering, from the analysis of the Jost ratio an so on [38].

In any case, the result (92) does not depend on our interpretation and thus we can write the toron measure in SQCD:

$$Z_{SQCD} = 2^5\pi^2\frac{1}{g^4}M_0^4 d^4x_0\frac{d^2\varepsilon}{M_0}\left(\frac{m}{M_0}\right)^{N_f/2}\exp\{-\frac{4\pi^2}{g^2}\}. \tag{95}$$

In obtaining (95) we take into account the equations (89), (92) and the expression for the toron measure for SYM (84) with constant C (88).

The expression (95) implies that the each fermion contribution equals $\sqrt{m}$ and so the formulation of the theory with $m = 0$ from the begining is in-correct. This fact was noticed in Refs. [28, 39], from another consideration. Thus, for the small (but non-zero) values of m, the theory is well defined and the chiral limit will be understood just in the sense of the limit, $m \to 0$.

8. Calculation of condensates in SQCD.

Because we have an explicit expression for the toron mesure (95), we can calculate the gluino condensate $\langle\lambda^2\rangle$ in SQCD. As usual, substituting in place of λ their zero modes (82), we verify, as in SYM, that $\langle\lambda^2\rangle$ is nonzero and equal to:

$$\langle g^2\lambda^2\rangle = 2^6\pi^3 \frac{1}{g^2} \Lambda_{1-loop}^{3-\frac{N_f}{2}} m^{\frac{N_f}{2}}, \quad \Lambda_{1-loop}^{3-\frac{N_f}{2}} \equiv M_0^{3-\frac{N_f}{2}} \exp\{-\frac{4\pi^2}{g^2}\} \tag{96}$$

As in the case of the calculation of $\langle\lambda^2\rangle$ in SYM (86), the expression (96) for $\langle\lambda^2\rangle$ in SQCD has precisely the renorminvariant form. Moreover, the value (96) we obtain for $\langle g^2\lambda^2\rangle$ satisfies the expected mass dependence ($\langle\lambda^2\rangle_{N_f=1} \sim \sqrt{m}$, $\langle\lambda^2\rangle_{N_f=2} \sim m$, [28]) and differs only by a numerical factor $\sqrt{4/5}$ from the value obtained in the Refs.[28, 39] and coinsides with the result of the Ref. [40].

Let us note, that the possibility of instanton calculations [40] is limited to the cases with $N_f < N = 2$. Our result (96) is still valid for all N_f and in this sense is a new one. It is more important, however, that in our approuch we can find the $\langle\lambda^2\rangle$ itself; in instanton calculations the nonzero contribution can be obtained only for the some Green function. The condensates in this approuch are obtained by extracting of the root. Besides that, the mass-dependence of (96) has a very unnatural form from the instanton point of view because instanton calculation gives an integer degree of (m) (the correct answer as mentioned above is obtained by exctracting of the root). In toron calculation the dependence $m^{1/2}$ is natural one, because in the toron back-ground the QZM

ensure exactly this dependence (92) of the toron mesuare (95).

In a more genearal case of the SU(N) gauge group one can think that the admissible value for the toron topological charge equals 1/N (see page.14.). In this case one can obtain that the dependence of $\langle\lambda^2\rangle$ on "m" is equals to $(m)^{N_f/N_c}$ in agreement with the general results of the supersymmetry and Ward identies [28].

The another important difference with the instanton calculation is related with the fact that in our approach, anaₗogously to the σ -model results, we have degenerate vacua (45) with spontaneously broken symmetry. The instanton calculation gives an average over these vacua [28] and we do not dwell on this issue.

At this point one can make use of the Konishi anomaly [41]

$$\frac{1}{32\pi^2}\langle g^2\lambda^2\rangle = m\langle\tilde{\varphi}\varphi\rangle \tag{97}$$

and find the value of the scalar field condensates. However, an independent determination of them is possible and will provide a valuable consistency check of our approach.

The computation of $\langle\tilde{\varphi}\varphi\rangle$ is more difficult problem, for two reason. The first is that $\langle\tilde{\varphi}\varphi\rangle$ is zero to lowest order in g because $\varphi = 0$ at the toron saddle point. In order to get a non-zero contribution to $\langle\tilde{\varphi}\varphi\rangle$ we have to bring down from $\exp\{-S_{Yukawa}\}$ the two vertices, see fig.6. Secondly, no closed form exists for the inverse of the operator $(D^2 - m^2)$ for nonvanishing masses in toron (like in instanton) background. This difficulty will be circumvented by going to a large m-limit, as it was discused in Refs [28, 39]. Using the method of these works, we obtain the following expression for the condensate:

$$\langle\tilde{\varphi}\varphi(x)\rangle = 2^5\pi^2 \frac{1}{g^4} \Lambda_{1-loop}^{3-\frac{N_f}{2}} m^{\frac{N_f}{2}} d^2\varepsilon(-2g^2m) \int d^4x_0\, d^4y\, d^4y' \cdot \tag{98}$$
$$\cdot\, tr\left(\frac{1}{D^2-m^2}\right)_{xy} \lambda_0(y)\left(\frac{1}{\bar{D}^2-m^2}\right)_{yy'} \lambda_0(y')\left(\frac{1}{D^2-m^2}\right)_{y'x}$$

In obtaining of the eq.(98) we take into account the expression (95) for the toron measure. Here λ_0 is the gluino zero modes (82), and $(D^2 - m^2)^{-1}$ is the correspomding Green function in the toron field , which locolized at the point x_0. More general case of the different values of the mass can be reached by the trivial replacement in the measure:

$$m^{N_f/2} \to \prod_{i=1}^{N_f} \sqrt{m_i}$$

Like in instanton calculations [28, 39] we can to replace the Green function $(D^2 - m^2)^{-1}$ by the free propagator in the limit $m \to \infty$. The evaluation of eq.(98) is now straightforward and gives:

$$\langle \tilde{\varphi}\varphi \rangle = 2^6 \pi^2 \Lambda^{3-\frac{N_f}{2}} m^{\frac{N_f}{2}} \frac{1}{m} \frac{1}{32\pi^2} \qquad (99)$$

in accordance with Konishi anomaly (97) and with the value of gluino condensate (96). The possibility of two independent calculations, hence, is the nontrivial check of our approach. It is not difficult to understand this result at least from a technical point of view. In fact the contribution surviving in the large m limit is identical to what one finds for the contribution of the regulators when the Pauli-Villars method is used in the deriviation of the anomaly equation.

Of course, thanks to supersymmetry, the $\langle \tilde{\varphi}\varphi \rangle$ condensate at the small m limit is the same, and equals to (99). However, an explicit check of the mass dependence of eq.(98) at $m \to 0$ is still lacking.

We end the discussion of the calculations in massive SQCD with a few remarks.

First, we expect that the integral (98) at small m-limit equals to the same value (99) and thus has a singularity (m^{-2}) at $m \to 0$. This means that QZM, playing an important role in the toron measure (see discussion of eq.(93)) should automatically give a correct expression for the integral (98) and correspondengly for the condensate (99).

As a second remark, we note that our results exhibit a discontinuous behaviour when going from the small mass limit of the massive theory to the strictly massless one from the begining. Indeed, at $N_f = 2$ we have the finite value for $\langle \tilde{\varphi}\varphi \rangle \sim \Lambda^2$ (99). Howevere, starting from the measure for the massless theory (in this case $Z_{SQCD}(m = 0) = 0$) we obtain the wrong result (we recall that the zero modes of quarks are absent in toron background). We think that the such behaviour is related with the fact that the massless theory is not well-defined.

9. The chiral condensate in QCD.

In the previous Section we saw that the spectrum of the Dirac operator for the fundamental representation of field is very unusual. Namely, the gap in the spectrum is absent and QZM to appear. Exactly these proporties ensure the correct results in the well-known supersymmetric theories. In particular, we saw the correctness mass dependence of condensates, the correctness renorm-invariant relation for the measure and so on. All these results is in agreement with the general relations of the supersymmetry, Ward identies and Konishi anomaly.

In this Section we shall discuss the toron calculation in the physically interesting theory of QCD with SU(2) gauge group. In this case, in compare with the toron measure in SQCD (95), we have the following distinctions:

a) the factor $d^2\varepsilon/M_0$ which is related with gluino zero modes is absent now;
b) the factor $(d_0)^{-N_f}$ relating with the scalar matter fields is also absent;
c) the non-zero modes are not cancel between bosons and fermions and should be taken into account.

Let us begin with the case of nonzero gauge modes. As is well known [42], up to logarithmic accuracy, the total contribution of the nonzero modes can be easily calculated with the help of the usual Feynman digrams, as was done for gauge theories in Ref.42 and for σ-model in Ref. [15]. The effective addition to the action is determined by fig. 7 and eguals [42]:

$$\begin{aligned} S &= S_{cl.} + \Delta S_g + \Delta S_f, \quad S_{cl.} = 4\pi^2/g^2(M_0) \\ \Delta S_g &= \frac{2}{3}\frac{g^2}{16\pi^2} \ln M_0 \int d^4x \left[\frac{1}{4}G_{\mu\nu}^2\right]_{cl.} = \frac{2}{3}(\ln M_0)Q \\ e^{-\Delta S_g} &= \exp\left(-\frac{2}{3}\cdot\frac{1}{2}\ln M_0\right) \end{aligned} \qquad (100)$$

By analogous way, we can find the effective addition to the action related with fermions. The result is determined by fig. 8 and equals [42]:

$$\Delta S_f = \frac{g^2 N_f}{16\pi^2} \ln M_0 \left(1 - \frac{1}{3}\right) \int d^4x \left[\frac{1}{4}G_{\mu\nu}^2\right]_{cl.} = QN_f(\ln M_0)\left(1-\frac{1}{3}\right) \qquad (101)$$

We have on purpose separated the fermion contribution (101) into two parts. One of them $QN_f(1)\cdot\ln M_0$ is related with the regulator contribution of zero modes at integer Q and with regulator contribution of QZM at fractional topological number in accordance with eq.(92). The second term in eq.(101) $(-\frac{1}{3}\ln M_0)\, QN_f$ is related with regulator contribution of nonzero modes and analogous to the contribution of the spinless field. This correspondence is in agreement with the expression (90), where the first term coincides with the spinless field contribution and the last term ensure the QZM contribution.

Collecting all factors together we obtain the following expression for the toron density in QCD:

$$Z_{QCD} = K\,\frac{M_0^4}{g^4}\,d^4x_0\left(\frac{m}{M_0}\right)^{N_f/2}\exp\left(-\frac{4\pi^2}{g^2}\right)\exp\left(-\frac{2-N_f}{6}\ln M_0\Delta\right) \tag{102}$$

Here the standard factor $M_0^4 g^{-4} d^4x_0$ is due to the four translation zero modes; the factor $(m/M_0)^{N_f/2}$ is connected with QZM; lastly, the factor $\exp\left(-\frac{2-N_f}{6}\ln M_0\Delta\right)$ is the contribution of gauge (100) and fermion (101) nonzero modes and "K" is the constant. In obtaining (102) we took into account the fact that the contribution of nonzero modes which related with the first term of eq.(90) is a small in the limit $m \to 0(\sim m^2\ln m$, see Appendix B). Let us recall that in SQCD this contribution cancels with the corresponding boson determinat d_B.

As it should be, the expression (102) for the toron measure has precisely the renorm-invariant form. It is easy to trace this phenomenon: while the action decreased by a factor two, the number of admissible zero modes decreased by the same factor and the contribution of nonzero modes (100, 101) is smaller by a factor of two also (because a factor Q).

The parameter Δ which is present in the expression (102) is the regulator of our toron solution. As discussed above it may be understood as the point defect with size $\Delta \to 0$. The important difference with the supersymmetric case is that in SUSY theories the regulator Δ in the expression for the toron measure is absent because of cancelation of nonzero mode contributions. In QCD case the dependence of toron density (102) on Δ appears.

In particular, for $N_f = 1$ we have $Z_{QCD}(N_f=1) \sim d^4x\,\Delta^{-1/6}$. It is mean that the toron density increas when regulator parameter Δ tends to zero. But semiclassical approximation which was used in obtaining of the expression (102) is correct only for the small densities ($Z \sim d^4x\,\Delta^{-1/6} \ll 1$). It is means that the semiclassical aproximation (or another words, dilute-gas approximation) is broke down in this case and the toron interactions should be taken into account.

This interpretation can be confirmed by the consideration of the two-toron system with the small separation. As was explained in the Section 6 this system can be understood as the instanton with the size $\rho = z_1 - z_2$ and with position $x_0 = \frac{1}{2}(z_1 + z_2)$. Now we shall find the toron interaction and verify that this interaction is very essential and it cancels the Δ-dependence of the density. The small toron separation arises in the place of Δ. Indeed, we define the toron interaction energy analogously to the instanton case [43] by the following way:

$$Z_{2\,torons} \sim d^4z_1\, d^4z_2\,\frac{1}{\Delta^{\frac{2-N}{6}}}\cdot\frac{1}{\Delta^{\frac{2-N}{6}}}\,e^{-W_{int.}} \tag{103}$$

Here z_1 and z_2 are the toron positions while the $\exp(-W_{int})$ took into account the toron interaction. The dilute - gas approximation for the two-toron contribution is simply the product of the two single toron densities (102), and it is independent of the toron separation.

We can find the $\exp(-W_{int.})$ now by subtracting the noninteracting two-toron contribution from the instanton one. Like in supersymmetris theories one can interpret $d^4x_0\, d^4\rho$ from the instanton measure as the integral over two toron collective coordinates (the translations z_1 and z_2). The interaction is defined only by the nonzero modes:

$$e^{-W_{int.}} = \exp\left\{-\left[\left(\frac{2-N_f}{3}\right)\ln M_0\rho - \left(\frac{2-N_f}{6}\right)\ln M_0\Delta - \left(\frac{2-N_f}{6}\right)\ln M_0\Delta\right]\right\} = \tag{104}$$
$$= \exp\left\{-\left(\frac{2-N_f}{6}\right)\ln\left|\frac{x_1-x_2}{\Delta}\right|\right\}.$$

Here the factor $\exp\left[-\left(\frac{2-N}{3}\right)\ln M_0\rho\right]$ is related with the instanton nonzero modes [32] and the two factors $\exp\left[-\left(\frac{2-N_f}{6}\right)\ln M_0\Delta\right]$ are related with the nonzero modes (102) of the two torons.

Substituting the eq.(104) in eq.(103) we verify that Δ-dependence disappear from the expression for density; in place of Δ arises the toron separation $|z_1 - z_2|$ accounting the interaction.

In particular at $N_f = 1$ we have

$$W_{int.}(z_1 \to z_2) = \frac{1}{3}\ln|z_1 - z_2| \to -\infty$$

It is means that there is a logarithmic attraction which gives an increase of the toron density. It is in agreement with the qualitative aforementioned notes. For $N_f \geqslant 3$, eq.(104) gives a logarithmic repulsion in the interaction energy*. This effect can be qualitatively understood as due to Fermi statistics.

The case of QCD with $N_f = 2$ (in a more general case $N_f = N_c$) calls for particular attention**. In this case the toron density (102) has finite limit at $\Delta \to 0$. From the technical point of view this singling out is related with the cancels of non-zero modes like in the supersymmetric models.

Summarising, we saw that to find the toron contribution to the different physical values in the general case does not possible in semiclassical approximation. This because dilute gas approximation is broke down and toron interaction should be taken into account. The toron individuality is lost in this case and they apperently "melt" like it was happened with the instantons in the O(3) σ model [44].

In the case with $N_f = 2$ (in general $N_f = N_c$) the toron preserves own individuality and can gives a finite contribution to the physical values. It so happens that this class of theories is very interesting from the physical point of view because in Nature we have $N_c = 3$ and N_f (the number of light flavors with $m = m_d = m_s = 0$) $= 3$. Here and in what follows we consider the case $N_f = N_c = 2$ only.

The toron measure (104) in this case takes the form:

$$Z_{QCD}(N_f = 2) = K \frac{M_0^4}{g^4} d^4x_0\, m \exp\left\{-\frac{4\pi^2}{g^2}\right\} = K m \Lambda^3 d^4 x_0$$

$$\Lambda^3 \equiv M_0^3 g^{-4} \exp\left\{-\frac{4\pi^2}{g^2}\right\} \tag{105}$$

We can find the constant " K" by the way which was described for the obtaining the constant "C" (88) in the supersymmetric

* The instanton interaction has the same qualitative proporties, in particular: $W_{inst.int.} \sim (2 - N_f)\ln|z_1 - z_2|$ [43].

** This point of view was intensively discussed early in Ref. [29].

theories. From (87), (105), [32] we obtain:

$$K = \pi^2 2^5 \left(\frac{e^{5/6}}{4}\right)^{1/2}. \tag{106}$$

We are now ready to calculate the chiral condensate in QCD. Like in SQCD the QZM play an important role in this calculation. Exactly these modes can to cancel the small mass factor $\sim m$ (105) out and to ensure the finite result in the limit $m \to 0$.

By defenition, with take into account eqs.(105,106) we have:

$$\langle \bar\psi \psi \rangle_{Minkowsky} = -i \langle \psi^+ \psi \rangle_{Euclid} =$$

$$= -i\pi^2 2^4 e^{5/12} m \cdot \Lambda^3 \cdot \int d^4x_0\, \mathrm{tr} \frac{-im}{[-\hat{D}^2 + m^2]} \tag{107}$$

Here ψ is the field of any light flavor (u, d). In obtaining (107) we substitute $\psi \psi^+$ by the Green function in the toron background. Let us note that the integral which was obtained in (107) was considered early when we calculated the fermion determinant in SQCD, see eq.(90). We don't know the massive Green function, but integrals li e (107) we know exactly in the limit $m \to 0$! Indeed,

$$\int d^4x_0\, \mathrm{tr} \frac{m^2}{(-\hat{D}^2 + m^2)} = \int d^4x_0\, \mathrm{tr}\left(\frac{m^2(1+\gamma_5)}{-D^2 + m^2}\right) - \int d^4x_0\, \mathrm{tr} \frac{m^2 \gamma_5}{-D^2 + m^2} = Q = \frac{1}{2} \tag{108}$$

Here, like in the eq.(90) we took into account the identity $\hat{D}^2(1+\gamma_5) = D^2(1+\gamma_5)$ which is c rrect one for any self-dual fields. Besides that we take into account that the first term of eq.(108) is a small in the limit $m \to 0(m^2 \ln m)$, see Appendix B, and the last term in (108) is related with the index of Dirac operator, actually independent of "m" and equals to topological charge $Q = 1/2$ of background field [36]. The antitoron gives the same contribution and so:

$$\langle \bar\psi \psi \rangle_{Minkowsky} = -\pi^2 2^4 \exp\left(\frac{5}{12}\right) \Lambda^3 \tag{109}$$

It is the main result of this work.

Like in supersymmetric theories the result (109) is defined by QZM and the chiral limit is understood only in the sense m $\to$ 0 of the massive theory.

As a second remark, we note that mechanism of spontaneous chiral symmetry breaking under consideration recalls the one of Ref.[45]. In both cases $\langle \bar{\psi}\psi \rangle \neq 0$ because of QZM at $\lambda \to 0$. The difference is that now the QZM are inherent property of configurations with fractional Q. In Ref. [45] this effect is due to the instanton interaction.

10. Scalar and pseudoscalar two-point correlation functions and the Goldstone theorem

In this Section we would like to check the Goldstone theorem saying that there should be light ($\sim$m) pseudoscalar boson if the chiral symmetry is spontaneously broken and condensate $\langle \bar{\psi}\psi \rangle$ is nonzero. Of course, we are not doubting a general theorem but rather we are eager to learn how it does work: What is the mechanism which ensures the singular behaviour of the pseudoscalar correlator (because of the light Goldstone boson) and does not exhibit the singular behaviour of the scalar one.

Besides that in the singlet pseudoscalar channel there should not be a Goldstone particle owing to the solution of the $U(1)$ problem (for a review see Ref.[12]). On the other hand, the $U(1)$ problem arises with condensate (109). So, in any consistent mechanism for chiral breaking the $U(1)$ problem must be solved in automatic way and pole in the corresponding singlet correlator must be absent.

With this discussion taken into account, we would like to consider the following three correlation functions:

$$P_5 \equiv i \int\limits_{q\to 0} dx\, e^{iqx} \langle 0| T\{\bar{u} i\gamma_5 d(x), \bar{d} i\gamma_5 u(0)\}|0\rangle \qquad (110)$$

$$P_{scalar} \equiv i \int\limits_{q\to 0} dx\, e^{iqx} \langle 0| T\{\bar{u} d(x), \bar{d} u(0)\}|0\rangle \qquad (111)$$

$$P_{singlet} \equiv i \int\limits_{q\to 0} dx\, e^{iqx} \langle 0| T\left\{\frac{\bar{u} i\gamma_5 u + \bar{d} i\gamma_5 d}{\sqrt{2}}(x), \frac{\bar{u} i\gamma_5 u + \bar{d} i\gamma_5 d}{\sqrt{2}}(0)\right\}|0\rangle \qquad (112)$$

(we work temporary in the Minkowsky space). As is well known (see e.g. Ref. [12]) for the pseudoscalar correlator P_5 there is a famous Ward identity:

$$P_5 = -\frac{\langle \bar{\psi}\psi \rangle}{m} \qquad (113)$$

which means that in this channel there should be light ($\sim$ m) Goldstone boson (π-meson) and this correlator should be tends to infinity when m $\to$ 0. The two another correlators (111), (112) should be regular ones. Now we would like to verify all these properties and the calculation of the corresponding correlators will provide a valuable consistency check of our approach.

Returning back to the Euclidean space and using the expression (105) for the toron measure we have (see Fig. 9)

$$P_5 = K m \Lambda^3 (\Delta_1 + \Delta_2), \quad K = \pi^2 e^{5/12} 2^4 \qquad (114)$$

$$\Delta_1 = T_z \frac{m}{-\hat{D}^2 + m^2} \frac{m}{-\hat{D}^2 + m^2}$$

$$\Delta_2 = T_z \overrightarrow{\hat{D}} \frac{1}{-\hat{D}^2 + m^2} \gamma_5 \frac{1}{-\hat{D}^2 + m^2} \overleftarrow{\hat{D}} \gamma_5$$

Here we employ an usual operator notation and the T_z arises because of factor d^4x_0 in the expression (105). Besides that we preserve the eigenfunctions with negative chirality $\gamma_5\psi_k = -\psi_k$ only because for the functions with positive chirality we have $\hat{D}^2(1+\gamma_5) = D^2(1+\gamma_5)$ and corresponding terms like in eqs.(90), (108) are the small ones in the limit m $\to$ 0. This fact is taken into account implicitly in relations (114). So, the corresponding terms here and in what follows will be omit because they can't to ensure the singular behaviour $\sim m^{-1}$ of the correlators.

The Δ_1-term in eq.(114) consists of the quark Green function propertional to m. This part of the correlator has the same sign for the scalar and pseudoscalar correlators because

$$\gamma_5 \frac{1}{\hat{D}^2 - m^2} \gamma_5 \frac{1}{\hat{D}^2 - m^2} = \frac{1}{\hat{D}^2 - m^2} \cdot \frac{1}{\hat{D}^2 - m^2}$$

The Δ_2 contribution consists of the quark Green function proportional to $D_\mu \gamma_\mu$ and so this part of the correlator has an opposite sign for the scalar and pseudoscalar correlators.

Now we will show that Δ_1 and Δ_2 contributions have the same values and tends to infinity like m^{-2} for $m \to 0$. In this case $P_5 \sim m^{-1}$ and $P_{scal.} \sim 1$ as it should be.

We don't know the closed form for the massive Green function. Fortunately, the evaluations of the integral (114) reduces to the well-known expression (108) (up to $m^2 \ln m \to 0$ accuracy), which is actually independent of m and equals 1/2 in the limit $m \to 0$.

To show this let us consider the relation (108) from the spectral representation point of view.

$$\lim_{m\to 0} \mathrm{Tr}\frac{m^2}{-\hat{D}^2+m^2} = \lim_{m\to 0}\mathrm{Tr}\sum_k \frac{m^2}{k^2+m^2}\psi_k(x)\psi_k^+(x) = Q \tag{115}$$

Here k^2 is the eigenvalue of the operator $\hat{D}^2$:

$$(-\hat{D}^2+m^2)\psi = (-D^2 + i\sigma_{\mu\nu}G^a_{\mu\nu}\frac{\tau^a}{2}+m^2)\psi = (k^2+m^2)\psi \tag{116}$$

Besides that it is easy to see that if φ is a scalar mode function, $(-D^2+m^2)\varphi = (k^2+m^2)\varphi$, then $\psi \sim \gamma_\mu D_\mu \varphi\varepsilon$ satisfies to the spinor mode equation (116) and its normalization differs from that of the scalar mode function φ by a factor of the eigenvalue (k^2+m^2). So, we have:

$$\mathrm{Tr}\frac{1}{-\hat{D}^2+m^2} = \mathrm{Tr}\sum_k \frac{\psi_k\psi_k^+}{k^2+m^2} = -2\,\mathrm{Tr}\,\vec{\hat{D}}_x\sum_k \frac{\varphi_k(x)\varphi_k^+(x')}{k^2+m^2}\overleftarrow{\hat{D}}_{x'} \tag{117}$$
$$= -2\,\mathrm{Tr}\,\vec{\hat{D}}\frac{1}{-D^2+m^2}\times\frac{1}{-D^2+m^2}\overleftarrow{\hat{D}}$$

In obtaining factor two in eq.(117) we took into account that for each solution of the scalar field equation there are precisely two linearly independent solutions of the spinor field equation because we have two orthogonal spinors ε_α. As we know the LHS of eq.(117) equals to Q/m^2 in the $m \to 0$ limit, see eq.(108). But the RHS of (117) is just what is needed for the calculation of the Δ_2 -contribution. Thus, from (114), (117) we have:

$$\Delta_2 = \frac{1}{2m^2}Q = \frac{1}{4m^2} \tag{118}$$

For the calculation of Δ_1 - contribution a little more work is needed. Let us rewritten the eq.(114) in the following form:

$$\Delta_1 = \mathrm{Tr}\sum_k \frac{m}{k^2+m^2}\cdot\frac{1}{2}\frac{m}{k^2+m^2}\psi_k(x)\psi_k^+(x) \equiv$$
$$\equiv \lim_{m\to m'}\frac{1}{2mm'}\mathrm{Tr}\sum_k\frac{m^2m'^2}{m^2-m'^2}\left(\frac{1}{k^2+m'^2}-\frac{1}{k^2+m^2}\right)\psi_k(x)\psi_k^+(x) =$$
$$= \lim_{m\to m'}\frac{1}{2mm'}\mathrm{Tr}\left[\frac{m^2}{m^2-m'^2}\left(\frac{m'^2}{-\hat{D}^2+m'^2}\right)-\frac{m'^2}{m^2-m'^2}\left(\frac{m^2}{-\hat{D}^2+m^2}\right)\right] \tag{119}$$

Here the factor 1/2 related with the fact that only functions with negative chirality give a main contribution ($\sim m^{-2}$) to the Δ_1. Besides that the expression (119) is understood in the limit sense ($m \to m'$). This formal manipulation of Δ_1 allows us to write the Δ_1 in terms of the index operator (108). From (108), (119) we obtain:

$$\Delta_1 = \lim_{m\to m'}\frac{1}{2mm'}\left(\frac{1}{2}\frac{m^2}{m^2-m'^2}-\frac{1}{2}\frac{m'^2}{m^2-m'^2}\right) = \frac{1}{4m^2} \tag{120}$$

As it should be $\Delta_1 = \Delta_2$ and correlator P_5 (114) equals (we took into account that the antitoron gives the same contribution to P_5):

$$P_5 = \pi^2 2^4 e^{\frac{5}{12}} m\Lambda^3\left(\frac{1}{4m^2}+\frac{1}{4m^2}\right)2 = \frac{\pi^2 2^4 e^{5/12}\Lambda^3}{m} \tag{121}$$
$$P_{scalar} = o\left(\frac{1}{m}\right)$$

which reproduces the result of the Ward identity (113) with the expression (109) for the chiral condensate.

We pass now to the analysis of the singlet channel. In this case besides of the diagram (fig.9) we should take into account the diagram fig.10. So, from (112) we have following expression for $P_{singlet}$ in the teron background:

$$P_{singlet} = P_5 - \frac{1}{2} K m \Lambda^3 T_r\left(\frac{2m\gamma_5}{-\hat{D}^2+m^2}\right) \cdot T_r\left(\frac{2m\gamma_5}{-\hat{D}^2+m^2}\right) \qquad (122)$$

Here the first term related with the standard contribution, of fig. 9. The last term (fig.10) corresponds to the factorization into a product of two averages. The sign minus due to Fermi statistics.

The toron contribution to P_5 was calculated before, the last term of (122) is exactly the square of the index operator (108). As usualy, preserving only a singular terms in the $P_{singlet}$, and using the eq.(114, 118,120) we have:

$$P_{singlet} = K\Lambda^3 \left(\frac{1}{2m}\right) - \frac{1}{2} K \Lambda^3 m \left(\frac{1}{m}\right)\left(\frac{1}{m}\right) = 0 \cdot \frac{1}{m} \qquad (123)$$

As it should be, the mass-singularity in the singlet channel is absent.

Formulae (121), (123) demonstrate the complete consistency of our understanding of the chiral symmetry breaking in the toron vacuum.

11. U(1) problem and θ-periodicity puzzle in QCD

In the previous Section we saw that in the singlet channel the mass - singularity is absent and so the U(1) "Goldstone boson" must have a finite mass in the chiral limit, $m \to 0$. But it is not a final of the story, because the solution of the U(1) problem means that the relevant anomalous Ward identities (WI) should be fulfiled [11,12]. In this Section we prove that the corresponding WI are satisfied in automatic way as it should be because the U(1) problem arises together with spontaneous breaking of chiral symmetry, i.e. with the condendate $\langle\bar{\Psi}\Psi\rangle$ (109).

Let us consider the following WI [11]:

$$i\int dx \left\langle T\left\{ \frac{g^2 G\tilde{G}}{32\pi^2}(x), \frac{g^2 G\tilde{G}}{32\pi^2}(0) \right\}\right\rangle = +\frac{1}{2} m \langle\bar{\Psi}\Psi\rangle \qquad (124)$$

$$i\int dx \left\langle T\left\{ \frac{g^2 G\tilde{G}}{32\pi^2}(x), \frac{\bar{u} i\gamma_5 u + \bar{d} i\gamma_5 d}{\sqrt{2}}(0) \right\}\right\rangle = -\frac{1}{\sqrt{2}} \langle\bar{\Psi}\Psi\rangle \qquad (125)$$

Like in previous section, we write WI in the Minkowsky space, in a standard notations. We took into account that the $P_{singlet}$ (123)-contribution is regular one and so the corresponding term was omited in WI.

As is well-known it is very difficult to satisfy the WI in the standard instanton picture (see Ref. [12] and references there in). For instance, the instanton dilute gas approximation shows [11,12] that the LHS of eq.(124) is of order m^2 and not of order m as RHS. Furthmore, the RHS depends on θ as $exp\{i\theta/2\}$ and it is hard to see low integer topological charge can give such dependence on θ.

The standard approach to this problem is related with assumption [46] that the LHS of eq.(124) is non-zero in pure YM theory. In this case, as was shown by Veneziano [47] in QCD with light flavour fields, the LHS should be order m^1 in agreement with WI. This approach is phenomenological one and can not to explain the dinamical questions such as: what kind of vacuum fluctuations are responsible for the solution of U(1) problem and corresponding WI.

Because for $N_f = 2$ $(N_f = N_c)$ we have some consistent mechanism for chiral breaking and can find the condensate $\langle\bar{\Psi}\Psi\rangle$ (109), the WI should be fulfiled by the same vacuum fluctuations. Indeed, returning back to the Euclidean space and using the expression (105) for the toron measure we have following expression for the LHS of eq.(124):

$$K m \Lambda^3 i^2 \int d^4x \frac{g^2 G\tilde{G}}{32\pi^2} \int d^4x_0 \frac{g^2 G\tilde{G}}{32\pi^2} + antitoron = -K m \Lambda^3 \left(\frac{1}{4}+\frac{1}{4}\right) \qquad (126)$$

in agreement with WI (124) and with value of condensate (109). Here the sign minus due to Euclidean space (remind $G\tilde{G}_M \to i G\tilde{G}_E$). It is absolutely crucial that correlator (124) is negative (in our notations); otherwise phenomenological formula [47] would result in the wrong sign mass^2.

In analogous way we have following expression for the LHS of eq.(125):

$$K m \Lambda^3 \int d^4x \frac{g^2 G\tilde{G}}{32\pi^2} \int d^4x_0 \, t_r \frac{-m\sqrt{2}\gamma_5}{-\hat{D}^2+m^2} + antitoron = \frac{1}{\sqrt{2}} K \Lambda^3 \qquad (127)$$

in agreement with WI (125). Like in the previous case the calculation corresponds to the factorization into a product of two averages. In obtaining of eq.(127) we took into account the expression (108) for the index operator.

We pass now to the analysis of θ-dependences of condensates. We start from the following WI [11,12,47]:

$$\langle \bar{\psi}_L \psi_R \rangle_\theta = \exp\left(-\frac{i\theta}{2}\right) \langle \bar{\psi}_L \psi_R \rangle_{\theta=0} \qquad (128)$$

We differentiate the LHS of eq.(128) with respect to θ at $\theta = 0$ and obtain:

$$\frac{d}{d\theta} \langle \bar{\psi}_L \psi_R \rangle_{\theta=0} = -i \int dx \left\langle \frac{g^2 G\tilde{G}}{32\pi^2}(x),\ \bar{\psi}_L \psi_R \right\rangle_{\theta=0} = -\frac{i}{2} \langle \bar{\psi}_L \psi_R \rangle_{\theta=0} \qquad (129)$$

In the last step we returning back to the Euclidean space and carry out the standard toron calculation as was discussed before. Here the factor (1/2) has a clear meaning. Indeed because the calculation corresponds to the factorization into a product of two averages, the terms $\int dx \frac{g^2 G\tilde{G}}{32\pi^2}$ gives an exactly the topological charge (1/2) of the toron. It is a general result: when we differentiate the LHS of eq.(128) n-times with respect to θ at $\theta = 0$ we obtain factor

$$\left(-i \int dx \frac{g^2 G\tilde{G}}{32\pi^2}\right)^n = \left(-\frac{i}{2}\right)^n \qquad (130)$$

in agreement with WI (128). In analogous way we can show that θ-dependence of the $\langle G\tilde{G} \rangle_\theta$ is equal to:

$$\left\langle \frac{g^2 G\tilde{G}}{32\pi^2} \right\rangle_\theta = -\sin\frac{\theta}{2} \langle \bar{\psi}\psi \rangle_{\theta=0} \cdot m \qquad (131)$$

in agreement with Veneziano approach [47]. We should emphasize that (non-trivial) θ-dependence is through (θ/2). An analogous situation has been discussed in σ-mode in (49) and in SYM in Ref.[10]. This general consequence of WI can be easily understood from dinamical point of view. Indeed, each differentiation with respect to θ gives the factor (130), which in the toron background with fractional topological number is just

what is needed.

We end the discussion of the calculations in QCD with a some remark. We saw that the very unusual but well-known proporties of QCD (like θ-dependence, U(1)-problem ,the Goldstone theorem, the counting of the discrete number of vacuum states at a fixed value of θ* and so on) are fulfiled in a very simple manner. On the other hand it is very difficult to satisfy all these properties in another approaches. From my point of view the any consistent mechanism for chiral symmetry breaking should be ensure all these properties in authomatic way, because they are strong interconnected to each others. So, our valuable check of these properties confirms the consistency of the approach described above.

12. Conclusion

The main point of this work is an analysis of the physical consenquences of the existence of fractional charge in the σ-models and, espesially, in the physically interesting theory, QCD. It is shown that the corresponding fluctuations ensure spontaneous breaking of the chiral symmetry and give a nonzero contribution to the chiral condensate.

It is very important, that the toron solution is determined on the manifold with boundary. In this case many questions arise such as: global boundary conditions, the stability of the solution, self-adjointness of Dirac operator, single-valuedness of the physical values and so on. These questions are interconnected and turn out to be self consistent only for the special choice of the topological number (Q=1/2 for SU(2)). Only for this value Q the correct renorminvariant dependence is restored.

Besides that, it is shown that in the Dirac's spectrum of the quarks, the gap between zero and the continuum is absent. This property plays a very important role in the forming of

---

* Remind, that this number, in a general equals N_f. In our case N_f=2 and these states analogous two vacua which was discussed in σ model (45) and in SYM and related with the discrete chiral symmetry breaking of the $U(1)_A$ - symmetry: $\psi \to \exp(i\frac{\pi}{2}\gamma_5)\psi$. We do not dwell on this issue.

the chiral condensates in SQCD and QCD. Exactly this property ensures the correct dependence of condensates on "m", "g", "θ" in the well-understood model SQCD and gives exactly two vacua (for SU(2)) in agreement with general results. The instanton calculation gives only an average over these vacua.

In QCD this property ensures the nonzero value of $\langle\bar\Psi\Psi\rangle$ and the singular behaviour of the pseudoscalar correlator (becasue of the light π -meson). Moreover, because the U(1) problem arises together with spontaneous breaking of chiral symmetry, the corresponding WI should be satisfied in an automatic way. We have demonstrated this. Bisedes that, we have checked that the θ-dependence of condensates agrees with WI (so-called θ-puzzle).

All these results confirm the consistency of our approach. From my point of view this description does not contradict the old idea of Ref.[48], according to which the instanton is the superposition of two objects with half -integer topological charge. In Ref.[48] such an object with Q = 1/2 was the meron [49], possessing infinite action. In a certain sense our solution is similar to the meron: both have zero size. There is also a difference: the toron has finite action, the meron infinite. This interpretation can be confirmed by consideration of arbitrary gauge group G in other theories, such as SYM, SQCD. In this case this conjecture is in agreement with numbers of the vacuum states and zero modes. Besides that, this conjecture ensures the correct renorminvariant dependence of the toron measure and the correct form of the axial anomaly.

In conclusion the author expresses gratitude to V.L.Chernyak, D.I.Dyakonov, A.Yu.Morozov, V.Yu.Petrov, A.Rosly, M.A. Shifman and A.I.Vainshtein for useful discussions and critical remarks.

Appendix B

The goal of this Appendix is to find the Green function in the toron background and to calculate the value

$$m^2\int d^4x\, tr\left(\frac{1}{-D^2+m^2}-\frac{1}{-\partial^2+m^2}\right) \qquad \text{(B1)}$$

at m → 0. As discussed above, see eq.(108), this value related with calculation of the toron measure (102) and chiral condensate in QCD. We shall show that the value (B1) tends to zero in the chiral limit and so the corresponding contribution in eq.(108) can be omited.

We start from the following representation of the toron solution (65) in the superpotential form:

$$A_\mu^a=-\bar\eta^a_{\mu\nu}\partial_\nu \ln\rho\,,\quad \rho=\frac{1}{2z}\left(G+\bar G\right)$$

$$G=\frac{i-g}{i+g}\qquad g(z)=\left(\frac{1-z}{1+z}\right)^{3/2}\quad z=r+it \qquad \text{(B2)}$$

Here G(Z) is the analytic function and ρ is the superpotential satisfying to the equation $\Box\rho = 0$(70). The form (B2) is just which was used in the instanton case [50]. So we can to use all the methods of this paper. We will look for, with Ref.[50], the solution of equation

$$-D_x^2\,\Delta(x,y)=\delta^4(x-y),\quad D_\mu=\partial_\mu+iA_\mu^a\frac{\tau^a}{2} \qquad \text{(B3)}$$

in the form:

$$\Delta(x,y)=\rho^{-1/2}\,\frac{F(x,y)}{4\pi^2(x-y)^2}\,\rho^{-1/2}(y) \qquad \text{(B4)}$$

where the function $F(x,y)$ must obey the condition

$$F(x,x)=\rho(x)$$

This because the propagation function must have the same short--distance singularity as the free propagation function,

$$\Delta(x \to y) = \frac{1}{4\pi^2 (x-y)^2}$$

Inserting the decomposition (B4) into the Green's function equation (B3) and using the facts that

$$\rho^{-1/2} \delta(x-y) \rho^{1/2}(y) = \delta(x-y) \quad , \quad \sigma_\mu^{\pm} = (\pm i; \vec{\sigma})$$

$$\sigma_\mu^- \partial_\mu \frac{\sigma_\mu^+ (x-y)_\mu}{2\pi^2 [(x-y)^2]^2} = \delta(x-y) \quad , \quad \sigma_\mu^- \partial_\mu \sigma_\nu^+ \partial_\nu = \partial_\lambda^2$$

we can to write the eq.(B3) in the following form

$$\sigma_\mu^+ \partial_\mu F(x,y) - \frac{2\sigma_\mu^+ (x-y)_\mu}{(x-y)^2} \left[\bar{F}(x,y) - \rho(x) \right] = 0 \qquad \text{(B5)}$$

To find the solution of this equation let us introduce the complex notations:

$$x_\mu = (t_1, \vec{n}_1 r_1), \; z_1 = r_1 + i t_1, \; \bar{z}_1 = r_1 - i t_1, \; G_1 = G(z_1).$$

$$\text{(B6)}$$

$$y_\mu = (t_2, \vec{n}_2 r_2), \; z_2 = r_2 + i t_2, \; \bar{z}_2 = r_2 - i t_2, \; G_2 = G(z_2)$$

It is easy to assert that eq.(B5) is satisfied if $F(x,y)$ is given by:

$$F(x,y) = \frac{1}{4} \left\{ \frac{(1+\bar{\sigma}\bar{n}_1)(1+\bar{\sigma}\bar{n}_2)}{z_1 + \bar{z}_2} (G_1 + \bar{G}_2) - \frac{(1-\bar{\sigma}\bar{n}_1)(1+\bar{\sigma}\bar{n}_2)}{\bar{z}_2 - \bar{z}_1} (\bar{G}_1 - \bar{G}_2) - \right.$$

$$\text{(B7)}$$

$$\left. - \frac{(1+\bar{\sigma}\bar{n}_1)(1-\bar{\sigma}\bar{n}_2)}{z_2 - z_1} (G_1 - G_2) + \frac{(1-\bar{\sigma}\bar{n}_1)(1-\bar{\sigma}\bar{n}_2)}{z_2 + \bar{z}_1} (G_2 + \bar{G}_1) \right\};$$

The proof rests on the use of proporties of proectional operators ($1 \pm \bar{\sigma}\bar{n}$) and analytical properties of function G(Z).

In particular, for instanton G(Z) equals $z + z^{-1}$ and formula (B7) pass to the well-known instanton solution. Now we are ready to calculate the integral (B1). The leading term of this trace for small m can be calculated from eqs.(A4, A7). From these equations one finds that $\mathrm{Tr}\left(-\frac{1}{D^2} + \frac{1}{\partial^2}\right)$ is logarithmically divergent at large x . A natural cutoff at the corresponding trace is provided by the mass m. So, we can find the integral (B1) with logarithmic accuracy by cutting the integral off from above by the value m^{-1} and using the masless Green function. Take into account the asymptotic behaviour of Green function from (B2), (B7):

$$G(z \to \infty) \to \frac{2}{3} + \frac{1}{2z} \quad , \quad \rho(z \to \infty) \to \frac{2}{3}$$

$$F(x,y) \to \frac{2}{3} + \frac{1}{2} (\sigma_\mu^- x_\mu)(\sigma_\nu^+ x_\nu) \frac{1}{x^2 y^2}$$

we have at m $\to$ 0:

$$m^2 \int_0^{1/m} d^4x \; \mathrm{tr} \left[\frac{1}{-D^2} - \frac{1}{4\pi^2 (x-y)^2} \right] \sim \frac{3}{2} m^2 \ln m \qquad \text{(B8)}$$

The result (B8) means that the corresponding contributions in toron measure and in the expression for chiral condensate can be omited in the limit m $\to$ 0.

R e f e r e n c e s

31. E.Witten, Phys. Rev. Lett. 38 (1977), 121.
32. G.'t Hooft, Phys. Rev. D14 (1976), 3432.
33. A.Actor, Rev. Mod. Phys. 51 (1979), 461.
34. L.S.Brown et al. Phys. Rev. D16 (1977), 417.
35. R.D.Carlitz, D.B.Creamer, Annal. of Phys. 118 (1979), 429.
36. R.Jackiw, C.Rebbi, Phys. Rev. D16 (1977), 1052.
37. J.Kiskis, Phys. Rev. D15 (1977), 2329.
38. D.Boyanovsky, R.Blankenbecler, Phys. Rev. D31 (1985),3234.
39. D.Amati, G.Rossi, G.Veneziano, Nucl. Phys. B249 (1985), 1; B263 (1986), 591.
40. V.A.Novikov et al., Nucl. Phys. B260 (1985), 157.
41. K.Konishi, Phys. Lett. B135 (1984), 439.
42. A.I.Vainshtein et al., Usp. Fiz. Nauk, 136 (1982), 554.
43. L.S.Brown, Creamer D.B., Phys. Rev. D18 (1978), 3695.
44. V.A.Fateev, I.V.Frolov, A.S.Schwartz, Nucl. Phys. B154 (1979), 1.
45. D.I.Dyakonov, V.Yu.Petrov, Nucl. Phys. B272 (1986), 475.
46. E.Witten, Nucl. Phys. B156 (1979), 269.
47. G.Veneziano, Nucl. Phys. B159 (1979), 213.
48. C.G.Callan, R.Dashen, D.Gross, Phys. Rev. D17 (1978), 2717.
49. V.De Alfaro, S.Fubini, G.Furlan, Phys. Lett. B65 (1977), 1631.
50. L.S.Brown et al., Phys. Rev. D17 (1978), 1583.

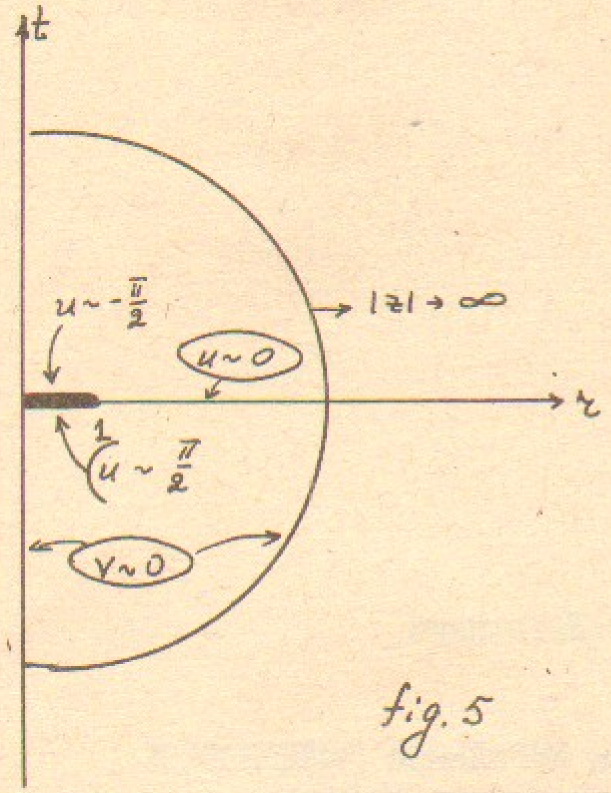

fig. 5

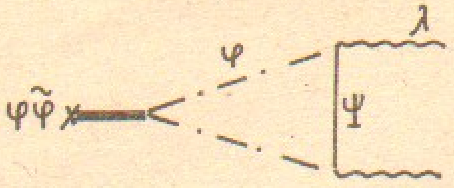

fig. 6

fig. 7

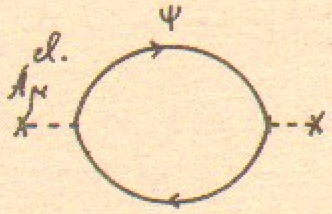

fig. 8

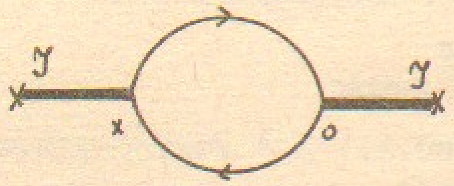

fig. 9

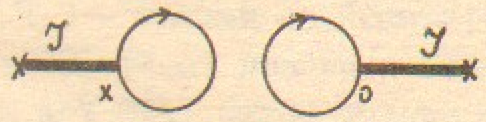

fig. 10

—·—·— scalar φ ~~~~~ gluino λ

——— quark ψ - - - - - gluon a_μ

x external field

А.Р.Житницкий

ТОРОНЫ И НАРУШЕНИЕ КИРАЛЬНОЙ СИММЕТРИИ В
σ-МОДЕЛЯХ И КАЛИБРОВОЧНЫХ ТЕОРИЯХ.
П. КАЛИБРОВОЧНЫЕ ТЕОРИИ.

Препринт
№ 89-I48

Работа поступила - 27 сентября I989г.

Ответственный за выпуск - С.Г.Попов
Подписано к печати I5.XI.I989г. МН I2I95
Формат бумаги 60х90 I/I6 Усл.2,9 печ.л., 2,4 учетно-изд.л.
Тираж 200 экз. Бесплатно. Заказ № I48.

Ротапринт ИЯФ СО АН СССР, г.Новосибирск, 90