ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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INVERSE SPECTRAL TRANSFORM FOR NONLINEAR EVOLUTION EQUATION GENERATING THE DAVEY — STEWARTSON AND ISHIMORI EQUATIONS

PREPRINT 89-123

НОВОСИБИРСК

Inverse Spectral Transform
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ABSTRACT

A (2+1)-dimensional nonlinear differential equation integrable by the inverse spectral transform method with the quartet operator representation is proposed. This GL (2, C)-valued chiral fields type equation is the generating (prototype) equation for the Davey — Stewartson and Ishimori equations. It coincides with the nonlinear equation for the Davey — Stewartson's eigenfunction Ψ_{DS}. The initial value problem for this equation is solved by the $\bar{\partial}$- and nonlocal Riemann — Hilbert problems method. The classes of exact solutions with the functional parameters and exponential-rational solutions are constructed by $\bar{\partial}$-dressing method. The static lump solution in the case $\alpha=i$ and exponentially localized solution at $\alpha=1$ are found. Other similar examples of nonlinear integrable equations in 2+1 and 1+1 dimensions are discussed.

1. INTRODUCTION

The inverse spectral transform (IST) method is one of the most powerful and effective method of the investigation of the nonlinear partial differential equations (see e. g. [1—6]). During last few years this method has been successfully generalized to the (2+1)-dimensional case (two spatial and one temporal coordinates). The discovery of the nonlocal Riemann — Hilbert problem and $\bar{\partial}$-problem method [7—9] has allowed to solve the initial value problems for a number of the nonlinear evolution systems in 2+1 dimensions. Among these systems are the Kadomtsev — Petviashvili (KP) equation [7, 10], the Davey — Stewartson (DS) equation [11—14], the Nizhnik — Veselov — Novikov equation [15, 16], the Ishimori equation [17, 18] and others (see e. g. reviews [19—21]). General version of the dressing method based on the use of the nonlocal $\bar{\partial}$-problem has been proposed in [22—24]. Recently the exponentially localized solitons for the DS-I equation have been found [25] and their spectral interpretations have been given in [26, 27]. In general, the (2+1)-dimensional IST method is now in an essential progress.

In the present paper we will study the nonlinear evolution equation

$$ig_t-\sigma_3(g_{xx}+\alpha^2 g_{yy})+\sigma_3 g_x g^{-1} g_x+\alpha^2\sigma_3 g_y g^{-1} g_y-$$
$$-\alpha g_y g^{-1} g_x-\alpha g_x g^{-1} g_y+(\alpha C_y+\sigma_3 C_x)\, g=0,$$
$$\alpha C_y-\sigma_3 C_x-\sigma_3(\alpha g_y g^{-1}-\sigma_3 g_x g^{-1})^2=0\,, \qquad (1.1)$$

where $g(x, y, t)$ is the invertable 2×2 matrix, $C(x, y, t)$ is the diagonal 2×2 matrix $f_y\equiv\frac{\partial f}{\partial y}$, $f_x\equiv\frac{\partial f}{\partial x}$, $f_t\equiv\frac{\partial f}{\partial t}$, $\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\alpha^2=\pm 1$ and $((\sigma_3 g_x-\alpha g_y)g^{-1})_{diag}=0$. It is shown that equation (1.1) is integrable by the IST method with the help of the auxiliary linear system

$$L_1\Psi\overset{\text{def}}{=}\left(\alpha g\frac{\partial}{\partial y}-\sigma_3 g\frac{\partial}{\partial x}\right)\Psi=0, \tag{1.2a}$$

$$L_2\Psi\overset{\text{def}}{=}\left(ig\frac{\partial}{\partial t}-2\alpha g\frac{\partial^2}{\partial x\,\partial y}-2\alpha g_x\frac{\partial}{\partial y}-2\alpha g_y\frac{\partial}{\partial x}\right)\Psi=0. \tag{1.2b}$$

The operator form of equation (1.1) (i. e. the operator form of the compatibility condition for the system (1.2)) is the quartet representation

$$[L_1, L_2]=\gamma_1 L_1+\gamma_2 L_2, \tag{1.3}$$

where

$$\gamma_1=2\alpha g_y\frac{\partial}{\partial x}+2\alpha g_x\frac{\partial}{\partial y}+a,$$

$$\gamma_2=[g,\sigma_3]\frac{\partial}{\partial x}+g(\alpha g_y-\sigma_3 g_x)g^{-1} \tag{1.4}$$

and

$$a=g\sigma_3(g_x g^{-1})_x-\alpha^2 g\sigma_3(g_y g^{-1})_y+$$
$$+\alpha g[g_y g^{-1}, g_x g^{-1}]+g(\alpha C_y+\sigma_3 C_x). \tag{1.5}$$

In the paper the initial value problem for equation (1.1) is solved for the class of solutions with asymptotic behaviour

$$g(x,y,t)\underset{x,y\to\infty}{\longrightarrow}1.$$

The corresponding inverse problem equations are generated by the $\bar\partial$-problem in the case $\alpha=i$ and by the nonlocal Riemann—Hilbert problem in the case $\alpha=1$.

Equation (1.1) is considered also within the framework of the $\bar\partial$-dressing method. The classes of the exact explicit solutions of equation (1.1) with functional parameters and rational-exponential solutions are constructed. Static lump (rational nonsingular) solution of equation (1.1) with $\alpha=i$ and the exponential solutions in the case $\alpha=1$ are also found.

Equation (1.1) is, in a certain sence, a generating (prototype)

equation for the DS and Ishimori equations. Namely, if $g(x, y, t)$ is a solution of equation (1.1) then the variables $S=-g^{-1}\sigma_3 g$ and

$$P=(\sigma_3 g_x-\alpha g_y)g^{-1}=\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}$$

obey the Ishimori and DS equations respectively. So, the solutions of equation (1.1) generate the corresponding solutions of the DS and Ishimori equations. In particular, the static lump solution of equation (1.1) with $\alpha=i$ gives rise to the vortex solution of the Ishimori-I equation and to the static lump solution of the DS-II equation. The exponential solutions of equation (1.1) at $\alpha=1$ (breather) generate the exponential localized solitons of the DS-I and Ishimori-II equations. The interrelation between equation (1.1) and DS equation can be treated as a semi-gauge equivalence while the Ishimori and DS equations, as known [17, 28], are gauge equivalent to each other.

Another important feature of equation (1.1) is that this equation exactly coincides with the nonlinear equation for the 2×2 matrix eigenfunction Ψ_{DS}, which corresponds to the DS equation. This equation for Ψ_{DS} arises after the exclusion of the potential P from the corresponding auxiliary linear system $L_{1DS}(P)\Psi_{DS}=0$, $L_{2DS}(P)\Psi_{DS}=0$. So, the nonlinear equation for the Davey—Stewartsons eigenfunction Ψ_{DS} is itself integrable by the IST method. The coincidence of equation (1.1) and the nonlinear equation for Ψ_{DS} is not accidential and it is connected with the semi-gauge equivalence of the DS equation and equation (1.1).

Equation (1.1) for the GL(2, C)-valued field $g(x, y, t)$ is, in fact, the principal chiral fields type equation. This equation can be rewritten in the equivalent form (see (2.1)) which includes only the left currents $g_t g^{-1}$, $g_x g^{-1}$ and $g_y g^{-1}$ and, hence, it has a simple algebraic and geometric sence.

Equation (1.1) and its interrelation with the nonlinear equation for eigenfunction Ψ_{DS} is not an exceptional case. It exemplifies a rather general phenomenon. In the present paper we discuss the KP equation and the resonantly interacting waves equation in the 2+1 dimensions from this point of view.

In the one-dimensional limit $g_y\equiv 0$ the system (1.1) is reduced to the simple local equation

$$ig_t-\sigma_3 g_{xx}+2\sigma_3 g_x g^{-1}g_x=0. \tag{1.6}$$

Equation (1.6) is the generating equation for the nonlinear Schrödinger (NLS) equation [1—6] and for the Heisenberg ferromagnet model equation [1—6]. Correspondingly equation (1.6) coincides with the nonlinear equation for the fundamental eigenfunction Ψ_{NLS} of the NLS equation. Similar results for the Korteweg-de Vries equation are also discussed.

The paper is organized as follows. In the second section the derivation of equation (1.1), its operator representations, the interrelation between equation (1.1) and DS, Ishimori and nonlinear equation for Ψ_{DS} are considered. In the sections 3 and 4 the solutions of the initial value problems for equation (1.1) are given in the cases $\alpha=i$ and $\alpha=1$. The $\bar{\partial}$-dressing method is applied to equation (1.1) in the section 5. The classes of the exact solutions with functional parameters and rational-exponential solutions are constructed. In the next section the static lump solution of equation (1.1) at $\alpha=i$ and exponential solutions in the case $\alpha=1$ are found. In the section 7 some other examples of the generating equations and equations for eigenfunctions in 2+1 and 1+1 dimensions are considered.

2. CHIRAL FIELDS TYPE PROTO-EQUATION: INTERPRETATION, DERIVATION, OPERATOR REPRESENTATIONS AND ALL THAT

1. Equation (1.1) is the equation for the field $g(x, y, t)$ which takes its values in the local group GL(2, C). Such fields are referred as a principal chiral fields (see e. g. [1, 6]). It is easy to show that equation (1.1) is equivalent to the following

$$\begin{gathered} ig_t g^{-1}-\sigma_3(g_x g^{-1})_x-\alpha^2\sigma_3(g_y g^{-1})_y- \\ -\alpha g_y g^{-1} g_x g^{-1}-\alpha g_x g^{-1} g_y g^{-1}+\alpha C_y+\sigma_3 C_x=0, \\ \alpha C_y-\sigma_3 C_x-\sigma_3(\alpha g_y g^{-1}-\sigma_3 g_x g^{-1})^2=0. \end{gathered} \tag{2.1}$$

Equation (2.1) contains only the left currents $g_t g^{-1}$, $g_x g^{-1}$, $g_y g^{-1}$ which belong to the local Lie algebra gl(2, C) and therefore it has a pure algebraic sence. If one denotes

$$J_0=g_t g^{-1}, \quad J_1=g_x g^{-1}, \quad J_2=g_y g^{-1}$$

and $x_0\equiv t$, $x_1\equiv x$, $x_2\equiv y$ then equation (2.1) becomes

$$\begin{gathered} iJ_0-\sigma_3 J_{1x_1}-\alpha^2 J_{2x_2}-\alpha J_1 J_2-\alpha J_2 J_1+\alpha C_{x_2}+\sigma_3 C_{x_1}=0, \\ \alpha C_{x_2}-\sigma_3 C_{x_1}-\sigma_3(\alpha J_2-\sigma_3 J_1)^2=0, \end{gathered} \tag{2.2a}$$

$$\frac{\partial J_i}{\partial x_k}-\frac{\partial J_k}{\partial x_i}+[J_1, J_k]=0, \quad (i, k=0, 1, 2), \tag{2.2b}$$

$$\sigma_3(\alpha J_2-\sigma_3 J_1)+(\alpha J_2-\sigma_3 J_1)\,\sigma_3=0, \tag{2.2c}$$

where equation (2.2c) means that $(\alpha J_2-\sigma_3 J_1)_{diag}=0$. A representability in the current form is a characteristic feature of the principal chiral fields type equations (see e. g. [1, 6]).

Equation (2.1) (or (2.2)) is invariant under the arbitrary right shifts $g\rightarrow g'=gh_r$ where h_r is an arbitrary constant GL(2, C)-valued matrix, but it is invariant only under special left shifts $g\rightarrow g'=h_l g$ where h_l is an arbitrary diagonal 2×2 matrix. Recall that the (1+1)-dimensional principal chiral fields equations (see e. g. [1, 6]) are invariant both under arbitrary left and right shifts. Algebraic formulation (2.2) of equation (1.1) can be used for its Lagrangian and Hamiltonian treatment.

2. An equivalence of equation (1.1) to the compatibility condition for the linear system (1.2), namely, to the operator equation (1.3) is verified straightforwardly.

As it has been pointed out in [29] the operators L_1 and L_2 are defined for the integrable equation which possess the quartet operator representation (1.3) nonuniquelly but up to the transformations

$$L_i\rightarrow L_i'=\sum_{k=1}^{2} C_{ik} L_k, \quad N_i\rightarrow N_i'=\sum_{k=1}^{2} N_k \tilde{Q}_{ki}, \tag{2.3}$$

where $N_1=L_2+\gamma_1$, $N_2=\gamma_2-L_1$ and Q_{ik} and $\tilde{Q}_{ki}$ are arbitrary differential operators which obey the constraint $\sum_{k=1}^{2}\tilde{Q}_{lk}Q_{ki}=\delta_{il}$. So, the whole family of the operators L_1 and L_2 correspond to equation (1.1). The operators $\tilde{L}_1$ and $\tilde{L}_2$ of the form

$$\tilde{L}_1=\alpha g\partial_y-\sigma_3 g\partial_x, \tag{2.3a}$$

$$\tilde{L}_2=ig\partial_t-2\sigma_3 g\partial_x^2-2(\alpha g_y+\sigma_3 g_x)\,\partial_x \tag{2.3b}$$

are the representative of this family. Indeed, equation (1.1) is equivalent to the operator equation $[\tilde{L}_1, \tilde{L}_2]=\tilde{\gamma}, \tilde{L}_1+\tilde{\gamma}_2\tilde{L}_2$ where

$$\begin{gathered} \tilde{\gamma}_1=2[\sigma_3, g]\,\partial_x^2+b\partial_x+a, \\ \tilde{\gamma}_2=[g, \sigma_3]\,\partial_x+g\,(\alpha g_y-\sigma_3 g_x)\,g^{-1}, \end{gathered} \tag{2.4}$$

a is given by (1.5) and

$$b=2\alpha(g_y-gg_yg^{-1})+2(g\sigma_3g_xg^{-1}+\sigma_3g_x)\,.$$

It is not difficult to check that the operators $\tilde L_1$, $\tilde L_2$, $\tilde\gamma_1$, $\tilde\gamma_2$ are connected with the operators L_1, L_2, γ_1, γ_2 by the relation

$$\tilde L_1=L_1\,,\quad \tilde L_2=L_2+2\partial_x L\,,\quad \tilde\gamma_2=\gamma_2\,,$$
$$\tilde\gamma_1=\gamma_1+[L_1,2\partial_x]-2\gamma_2\partial_x\,. \tag{2.5}$$

The relation (2.5) is the special case of the transformation (2.3) with $Q_{ii}=\tilde Q_{ii}=1$ $(i=1,2)$, $Q_{12}=\tilde Q_{12}=0$ and $Q_{21}=-\tilde Q_{21}=2\partial_x$.

Note that the operator $\tilde L_2$ can be obtained from the operator L_2 simply by the exclusion of the derivative ∂_y from L_2 with the use of the equation $L_1\Psi=0$. Different pairs of the operators L_1 and L_2 can be convenient for the different purposes.

3. Here we will consider another way of derivation of equation (1.1). Let us start with the auxiliary linear system

$$L_1\Psi=(\alpha g\partial_y-\sigma_3g\partial_x)\Psi=0\,, \tag{2.6a}$$
$$L_2\Psi=(ig\partial_t+Q_1\partial_x\partial_y+Q_2\partial_x+Q_3\partial_y)\Psi=0\,, \tag{2.6b}$$

where Q_1, Q_2 and Q_3 are unknown 2×2 matrices and try to find a nonlinear equation for g which guarantee the compatibility of the system (2.6). Let us look for the solutions of the system (2.6) of the form

$$\Psi(x,y,t)=\chi(x,y,t,\lambda)\times$$
$$\times\begin{vmatrix}\exp\left\{\frac{i}{2\lambda}\left(y+\alpha x+\frac{\alpha^2}{\lambda}t\right)\right\}, & 0\\ 0, & \exp\left\{-\frac{i}{2\lambda}\left(y-\alpha x+\frac{\alpha^2}{\lambda}t\right)\right\}\end{vmatrix} \tag{2.7}$$

where λ is an arbitrary complex parameter and the 2×2 matrix function χ has the following properties:

$$\chi(x,y,t,\lambda)\to1 \quad \text{at} \quad \lambda\to\infty\,,$$
$$\chi=\chi_0(x,y,t)+\lambda\chi_1(x,y,t)+\dots \quad \text{at} \quad \lambda\to0\,. \tag{2.8}$$

In virtue of (2.6), the function χ obeys the system of equations

$$\alpha g\partial_y\chi-\sigma_3g\partial_x\chi-\frac{i\alpha}{2\lambda}[\sigma_3,g\chi]=0\,, \tag{2.9a}$$

$$ig\partial_t\chi+Q\partial_x\partial_y\chi+\left(\frac{i\alpha}{2\lambda}Q_1+Q_3\right)\partial_y\chi+Q_2\partial_x\chi+\frac{i}{2\lambda}Q_1\partial_x\chi\sigma_3+$$
$$+\frac{i\alpha}{2\lambda}Q_2\chi+\frac{i}{2\lambda}Q_3\chi\sigma_3-\frac{\alpha}{4\lambda^2}(2\alpha g+Q_1)\chi\sigma_3=0\,. \tag{2.9b}$$

The normalization $\chi\xrightarrow[\lambda\to\infty]{}1$ is obviously admitted by the system (2.9). Then the condition of the absence of singularities in equations (2.9a) at $\lambda\to0$ gives

$$[\sigma_3,g\chi_0]=0\,, \tag{2.10}$$

i. e.

$$g=A\chi_0^{-1}\,, \tag{2.11}$$

where $A(x,y,t)$ is an arbitrary diagonal matrix. So there exists a certain arbitrariness in the relation between g and χ_0. This freedom can be removed by the introduction of the field $\bar g$ such that

$$\bar g=\chi_0^{-1}\,. \tag{2.12}$$

The system (2.9) can be rewritten in the terms of $\bar g$ and $\bar Q_i=A^{-1}Q_i$ only. To achieve this it is sufficient to multiply equations (2.9) by A^{-1} from the left.

Further, the regularity conditions at $\lambda\to0$ for equation (2.9b) gives

$$\bar Q_1=-2\alpha\bar g\,,\quad \bar Q_2=-2\alpha\bar g_y\,,\quad \bar Q_3=-2\alpha\bar g_x\,. \tag{2.13}$$

As a result, we arrive at the system (1.2) with $g\to\bar g$ and at the following system for χ:

$$\alpha\bar g\partial_y\chi-\sigma_3\bar g\partial_x\chi-\frac{i\alpha}{2\lambda}[\sigma_3,\bar g\chi]=0\,, \tag{2.14a}$$
$$i\bar g\partial_t\chi-2\alpha\bar g\partial_x\partial_y\chi-2\alpha(\bar g_x\partial_y\chi+\bar g_y\partial_x\chi)-$$
$$-\frac{i\alpha}{\lambda}(\alpha\partial_y(\bar g\chi)+\partial_x(\bar g\chi)\sigma_3)=0\,. \tag{2.14b}$$

Now one should extract the nonlinear equation for g from the auxiliary system (2.14). To do this let us substitute the expansion

$$\chi(x,y,t,\lambda)=\chi_0(x,y,t)+\lambda\chi_1(x,y,t)+\lambda^2\chi_2(x,y,t)+\dots$$

into the system (2.14). Equation (2.14) gives rise to the condition

(2.10) and the recurrent relations

$$\alpha\bar{g}\partial_y\chi_0-\sigma_3\bar{g}\partial_x\chi_0=\frac{i\alpha}{2}[\sigma_3,\bar{g}\chi_1], \tag{2.15a}$$

$$\alpha\bar{g}\partial_y\chi_1-\sigma_3\bar{g}\partial_x\chi_1=\frac{i\alpha}{2}[\sigma_3,\bar{g}\chi_2]. \tag{2.15b}$$

The relation (2.15a) implies that the matrix $\alpha\bar{g}\partial_y\bar{g}^{-1}-\sigma_3\bar{g}\partial_x\bar{g}^{-1}$ is the off-diagonal one. In the term of g this means that $(\alpha g\partial_y(g^{-1}A)-\sigma_3 g\partial_x(g^{-1}A))_{diag}=0$. This condition fixes the matrix A. Namely, the matrix A in (2.11) should be choosen in such a way that

$$(\alpha(A^{-1}g)\,\partial_y(A^{-1}g)^{-1}-\sigma_3(A^{-1}g)\,\partial_x(A^{-1}g)^{-1})_{diag}=0, \tag{2.16}$$

or

$$(\alpha g\partial_y g^{-1}-\sigma_3 g\partial_x g^{-1})_{diag}+(\alpha\partial_y-\sigma_3\partial_x)\ln A=0. \tag{2.17}$$

It is clear from (2.17) that the above condition does not fix the matrix A uniquelly, but up to the multiplication by the matrix A_0 which obeys the equation $(\alpha\partial_y-\sigma_3\partial_x)\ln A_0=0$. Such a matrix is of the form

$$A_0=\begin{pmatrix} e^{\varphi_1(y+\alpha x,t)}, & 0 \\ 0, & e^{\varphi_2(y-\alpha x,t)} \end{pmatrix} \tag{2.8}$$

where φ_1 and φ_2 are the arbitrary functions. Hence, one can introduce the variable $\bar{g}$ by the relation

$$\bar{g}\overset{def}{=}A_0\chi_0^{-1}=A_0\bar{g}, \tag{2.19}$$

where A_0 is an arbitrary matrix of the form (2.18) and rewrite all the equations in the terms of $\bar{g}$. These equations will contain the arbitrariness connected with the matrix A_0.

Other way is to formulate all relations and equations in the term of variable $\bar{g}$ defined by (2.12). In the further consideration we will follow to this way. We will omit the bar and will write simply g instead of $\bar{g}$.

Now let us return to equation (2.14b). The substitution of the expansion (2.8) into (2.14b) gives

$$ig\partial_t\chi_0-2\alpha g\partial_x\partial_y\chi_0-2\alpha(g_x\partial_y\chi_0+g_y\partial_x\chi_0)-$$

$$-i\alpha(\alpha\partial_y(g\chi_1)+\partial_x(g\chi_1)\,\sigma_3)=0, \tag{2.20a}$$

$$ig\partial_t\chi_1-2\alpha g\partial_x\partial_y\chi_1-2\alpha(g_x\partial_y\chi_1+g_y\partial_x\chi_1)-$$
$$-i\alpha(\alpha\partial_y(g\chi_2)+\partial_x(g\chi_2)\,\sigma_3)=0, \tag{2.20b}$$

. . . .

Since $\chi_0=g^{-1}$, then equation (2.20a) is, in fact, the nonlinear evolution equation for g. In order to rewrite (2.20a) in a closed form one should use the relations (2.15). The relation (2.15a) gives

$$i\alpha(g\chi_1)_{off}=\sigma_3(\sigma_3 g_x-\alpha g_y)\,g^{-1}, \tag{2.21}$$

where $\Phi_{off}\overset{def}{=}\Phi-\Phi_{diag}$, $(\Phi_{diag})_{ik}\overset{def}{=}\Phi_{ii}\delta_{ik}$ for an arbitrary matrix Φ. Then, the relation (2.15b) implies that $(\alpha\bar{g}\partial_y\chi_1-\sigma_3\bar{g}\partial_x\chi_1)_{diag}=0$. After some transformations this condition gives

$$i\alpha(\alpha\partial_y-\sigma_3\partial_x)\,(g\chi_1)_{diag}=(\alpha g_y g^{-1}-\sigma_3 g_x g^{-1})\,(g\chi_1)_{off}=$$
$$=\sigma_3(\alpha g_y g^{-1}-\sigma_3 g_x g^{-1})^2. \tag{2.22}$$

Denoting

$$C\overset{def}{=}i\alpha(g\chi_1)_{diag} \tag{2.23}$$

and substituting (2.12), (2.21) and (2.22) into (2.20a), we obtain the nonlinear evolution equation for $g(x,y,t)$ which coincides with equation (2.1) and, equivalently, equation (1.1).

In a similar manner one can construct equation (1.1) starting from the auxiliary system with the operators $\tilde{L}_1$ and $\tilde{L}_2$ of the form (2.3).

4. Now we will demonstrate the generating (proparent) character of equation (1.1) with respect to the DS and Ishimori equations.

At first, we recall the DS and Ishimori equations. The DS equation is (see e. g. [1—3])

$$iP_t-\sigma_3(P_{xx}+\alpha^2P_{yy})+\sigma_3P\,\mathrm{tr}\,(\sigma_3Q_D)=0,$$
$$(\alpha\partial_y-\sigma_3\partial_x)\,Q_D=\sigma_3(\alpha\partial_y+\sigma_3\partial_x)\,P^2, \tag{2.24}$$

where $P=\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}$ and Q_D is a diagonal matrix and $\alpha^2=\pm1$. The DS equation is the compatibility condition for the linear system

$$L_1^{DS}\Psi_{DS}\overset{def}{=}(\alpha\partial_y-\sigma_3\partial_x+P)\,\Psi_{DS}=0, \tag{2.25a}$$

$$L_2^{DS}\Psi_{DS}\overset{def}{=}(i\partial_t-2\sigma_3\partial_x^2+2P\partial_x+Q_D+\sigma_3(\alpha P_y+\sigma_3P_x))\,\Psi_{DS}=0. \tag{2.25b}$$

The operator form of the compatibility condition is $[L_1^{DS}, L_2^{DS}]=0$.

The Ishimori equation [30] looks like

$$\vec{S}_t(x,y,t)+\vec{S}\times(\vec{S}_{xx}+\alpha^2\vec{S}_{yy})+\Phi_x\vec{S}_y+\Phi_y\vec{S}_x=0\,,$$
$$\Phi_{xx}-\alpha^2\Phi_{yy}+2\alpha^2\vec{S}(\vec{S}_x\times\vec{S}_y)=0\,, \qquad (2.26)$$

where $\vec{S}=(S_1, S_2, S_3)$ is the unit vector $\vec{S}^2=1$, $\Phi(x, y, t)$ is a scalar function and $\alpha^2=\pm 1$. The Ishimori equation (2.26) is equivalent to the compatibility condition for the auxiliary system [30]

$$L_1^{Is}\Psi_{Is}\stackrel{def}{=}(\alpha\partial_y+S\partial_x)\Psi_{Is}=0\,,$$
$$L_2^{Is}\Psi_{Is}\stackrel{def}{=}(i\partial_t+2S\partial_x^2+(S_x+\alpha S_yS-i\alpha^3S\Phi_x+i\Phi_y)\,\partial_x)\,\Psi_{Is}=0\,, \qquad (2.27)$$

where $S=\vec{S}\vec{\sigma}$ and σ_1, σ_2, σ_3 are Pauli matrices. The operator form of equation (2.26) is $[L_1^{Is}, L_2^{Is}]=0$. Note that both the DS and Ishimori equations contain nonevolution variables Q_D and Φ.

The DS, Ishimori equations and equation (1.1) are similar in their forms and their close interrelation is not so suprising.

Let us start with equation (1.1) and DS equation. Using the condition $(\sigma_3g_xg^{-1}-\alpha g_yg^{-1})_{diag}=0$ or, equivalently,

$$\sigma_3(\sigma_3g_xg^{-1}-\alpha g_yg^{-1})+(\sigma_3g_xg^{-1}-\alpha g_yg^{-1})\,\sigma_3=0 \qquad (2.28)$$

it is not difficult to verify that if g obeys equation (1.1) then the combinations

$$P=\sigma_3g_xg^{-1}-\alpha g_yg^{-1}, \qquad (2.29)$$

$$Q_D=(\alpha\partial_y+\sigma_3\partial_x)\,C \qquad (2.30)$$

obey the DS equation (2.24). So any solution of equation (1.1) generates the solution of the DS equation by the formulae (2.29), (2.30).

The fact that the DS equation is a consequence of equation (1.1) can be proved in a different manner with the use directly of the auxiliary system (2.14).

Indeed, the relation (2.15a) gives (2.21) or

$$\sigma_3g_xg^{-1}-\alpha g_yg^{-1}=i\alpha\sigma_3(g\chi_1)_{off}\,. \qquad (2.31)$$

An evolution equation for the quantity $P\stackrel{def}{=}i\alpha\sigma_3(g\chi_1)_{off}$ follows directly from equation (2.20a) (or (1.1)) and equation (2.20b). One has

$$iP_t=-\frac{\alpha}{2}\,[\sigma_3, g_t\chi_1+g\chi_{1t}]=$$
$$=\sigma_3(P_{xx}+\alpha^2P_{yy})-\sigma_3P\,\mathrm{tr}\,((\alpha\sigma_3\partial_y+\partial_x)\,i\alpha(g\chi_1)_{diag})\,. \qquad (2.32)$$

Taking into account (2.22), we finally obtain DS equation (2.24) with

$$Q_D=(\alpha\partial_y+\sigma_3\partial_x)\,i\alpha\,(g\chi_1)_{diag}\,. \qquad (2.33)$$

So, the DS equation is nothing but the closed equation for $P\stackrel{def}{=}i\alpha\sigma_3(\chi_0^{-1}\chi_1)_{off}$, generating by the linear system (2.14). Both equation (1.1) for $g=\chi_0^{-1}$ and the DS equation for $P\sim\chi_0^{-1}\chi_1$ are the two first members of the family of nonlinear equations for χ_0, χ_1, χ_2, ... which follow from the linear system (2.14). Equation (1.1) is obviously the basic equation in this family. In virtue of the recurrent relations for χ_0, χ_1, χ_2, ... all other these equations are its consequences. In particular, equation (1.1) generates the DS equation by the transition to the variables (2.29) and (2.30).

Note that in such a treatment not only the potential P for the DS equation but also the nonevolution field Q_D are reconstructed through the eigenfunction χ:

$$P=i\alpha\sigma_3(g\chi_1)_{off}\,, \qquad Q_D=(\alpha\partial_y+\sigma_3\partial_x)\,i\alpha(g\chi_1)_{diag}\,. \qquad (2.34)$$

Recall that in all these formulas $g\equiv\tilde{g}=\chi_0^{-1}$. The corresponding equations written in the terms of $\tilde{g}=A_0\chi_0^{-1}$ will contain the arbitrary matrix A_0 of the form (2.18).

Now let us proceed to the Ishimori equation. It is not difficult to check that if g obeys equation (1.1) then the variable

$$S=-g^{-1}\sigma_3g \qquad (2.35)$$

obeys the Ishimori equation (2.26) with

$$\Phi(x,y,t)=2i\alpha\,\ln\det g\,. \qquad (2.36)$$

In the proof one should use the condition $(\sigma_3g_xg^{-1}-\alpha g_yg^{-1})_{diag}=0$. Then, since $L_1^{(1,2)}=gL_1^{Is}$, $L_2^{(1,2)}=gL_2^{Is}$, the auxiliary linear problem (1.2) is converted into the linear problem (1.2) is converted into the linear problem (2.27).

Note that the expression (2.36) for the scalar function Φ in the case $\alpha=i$ has been obtained for the first time within the direct linearizing transform approach in [28]. Note also that the term $\alpha C_y+\sigma_3C_x$ does not contribute into the quantity

$S_t=g^{-1}[g_tg^{-1},\sigma_3]g$ and that the second equation (2.26) is satisfied identically due to (2.36).

Thus, the solution g of equation (1.1) generates by the formulae (2.35) and (2.36) the solution $S=\vec{S}\vec{\sigma}$ and Φ of the Ishimori equation.

So, we see that equation (1.1) can be considered as the fundamental (prototype) equation with respect to the DS and Ishimori equations.

2. The coincidence of equation (1.1) with the nonlinear equation for the function Ψ_{DS} is one more important property of this equation.

Indeed, from equation (2.25a) one has

$$P=(\sigma_3\Psi_{DS_x}-\alpha\Psi_{DS_y})\Psi_{DS}^{-1}. \qquad (2.37)$$

Substituting this expression for P into equation (2.25b), we obtain

$$i\Psi_{DS_t}-\sigma_3(\Psi_{DS_{xx}}+\alpha^2\Psi_{DS_{yy}})+$$
$$+\sigma_3\Psi_{DS_x}\Psi_{DS}^{-1}\Psi_{DS_x}+\alpha^2\sigma_3\Psi_{DS_y}\Psi_{DS}^{-1}\Psi_{DS_y}-$$
$$-\alpha\Psi_{DS_x}\Psi_{DS}^{-1}\Psi_{DS_y}-\alpha\Psi_{DS_y}\Psi_{DS}^{-1}\Psi_{DS_x}+Q_D=0 \qquad (2.38)$$

that exactly coincides with equation (1.1) if one takes into account the second equation (2.24).

This fact demonstrates the fundamental character of equation (1.1).

The coincidence of equation (1.1) and equation (2.38) is not accidental and it is closely connected with the gauge equivalence of the linear systems (2.25) and (1.2). Indeed, let us perform in the system (2.25) the gauge transformation

$$\Psi_{DS}=G\tilde{\Psi}. \qquad (2.39)$$

As a result, the system (2.25) is transformed into the following one

$$\tilde{L}_1\tilde{\Psi}=(\alpha G\partial_y-\sigma_3G\partial_x+(\alpha G_y-\sigma_3G_x+P(G))\tilde{\Psi}=0,$$
$$\tilde{L}_2\tilde{\Psi}=(iG\partial_t-2\sigma_3G\partial_x^2-2(2\sigma_3G_x-PG)\partial_x+$$
$$+(-2\sigma_3G_{xx}+2PG_x+iG_t+Q_DG+(\alpha\sigma_3P_y+P_x)G)\tilde{\Psi}=0. \qquad (2.40)$$

Now let us choose G such that

$$\alpha G_y-\sigma_3G_x+PG=0,$$
$$iG_t-2\sigma_3G_{xx}+2PG_x+(Q_D+\alpha\sigma_3P_y+P_x)G=0. \qquad (2.41)$$

With such a G the system (2.40) becomes

$$\tilde{L}_1\tilde{\Psi}=(\alpha G\partial_y-\sigma_3G\partial_x)\tilde{\Psi}=0,$$
$$\tilde{L}_2\tilde{\Psi}=(iG\partial_t-2\sigma_3G\partial_x^2-2(\alpha G_y+\sigma_3G_x)\partial_x)\tilde{\Psi}=0. \qquad (2.42)$$

The operators $\tilde{L}_1$ and $\tilde{L}_2$ exactly coincide with the operators (1.2) and, hence, the variable G obeys equation (1.1). On the other hand, the variable G obeys the system (2.4) that is exactly the auxiliary linear system (2.25) for the DS equation.

So, each solution of equation (1.1) is the certain DS eigenfunction Ψ_{DS} and vice versa. Such a coincidence gives us also a way of constructing the solutions of equation (1.1) using the known DS eigenfunctions Ψ_{DS}.

The interrelation between the operators L_1^{DS}, L_2^{DS} and the operators L_1, L_2 given by (1.2) is a very simple one. Indeed, starting with the linear system (1.2) and performing the gauge transformation $\Psi=g^{-1}\tilde{\Psi}$, we arrive at the linear system for $\tilde{\Psi}$ which coincides with the system (2.25) with $P=\sigma_3g_xg^{-1}-\alpha g_yg^{-1}$. So

$$L_1^{DS}=L_1^{(1.2)}g^{-1},\quad L_2^{DS}=L_2^{(1.2)}g^{-1}. \qquad (2.43)$$

The relation (2.43) and correspondingly the interrelation between equation (1.1) and DS equation can be treated as a semi-gauge equivalence.

A usual gauge equivalence (see e. g. [1]) takes place between the Ishimori and DS equations [17, 28]:

$$L_1^{DS}=gL_1^{Is}g^{-1},\quad L_2^{DS}=gL_2^{Is}g^{-1}. \qquad (2.44)$$

The coincidence mentioned above can be treated also in a following way. Let one has the auxiliary linear system (2.25). The exclusion of the function Ψ_{DS} from this system gives rise to the DS equation for P which is integrable by the IST method. On the other hand, the exclusive of the potential P from the system (2.25) produces the nonlinear evolution equation for Ψ_{DS} which, as we have seen, is integrable by the IST method too. So, the DS equation for P and equation (1.1) for Ψ_{DS} can be treated as the irreducible integrable forms of the basic mixed (reducible) linear system (2.25).

Similar situation takes place for other integrable equations too. Some examples will be discussed in the section 7.

3. INITIAL VALUE PROBLEM: THE CASE $\alpha=i$

In this and next sections we will present the solution of the initial value problem for equation (1.1) for the class of solutions with the asymptotic behaviour $g(x, y, t)\to 1$ as $x^2+y^2\to\infty$.

The method of solution is the standard $\bar{\partial}$ and nonlocal Riemann—Hilbert problems method (shortly $\bar{\partial}$—NRGP method) and many formulae are similar to those found for the Ishimori equation [17]. By this reason we will omit some details.

We start with the case $\alpha=i$. It is convenient to introduce the complex variables

$$z=\frac{1}{2}(y+ix), \quad \bar{z}=\frac{1}{2}(y-ix).$$

The starting point is the problem (2.14a) which in our case can be written as follows

$$\begin{pmatrix}\partial_{\bar z} & 0\\ 0 & \partial_z\end{pmatrix}\chi-\frac{i}{2\lambda}[\sigma_3,\chi]-\frac{i}{2\lambda}[\sigma_3,(g-1)\chi]+$$
$$+\frac{1-\sigma_3}{2}(g-1)\,\partial_z\chi+\frac{1+\sigma_3}{2}(g-1)\,\partial_{\bar z}\chi=0. \qquad (3.1)$$

We firstly should solve the inverse problem for equation (3.1). Following to the standard $\bar{\partial}$—NRHP method, we will consider the solutions of equation (3.1) bounded for all λ (possibly with the finite number of singular points) and normalized by the condition $\chi\xrightarrow[\lambda\to\infty]{}1$. Such solutions of equation (3.1) obey also the integral equation

$$\chi(z,\bar z,\lambda,\bar\lambda)=1-(\hat{G}Q(\partial,\bar\partial,\lambda)\,\chi(\ldots,\lambda,\bar\lambda))\,(z,\bar z), \qquad (3.2)$$

where $\hat{G}$ is the operator inverse to the operator

$$L_0=\begin{pmatrix}\partial_{\bar z} & 0\\ 0 & \partial_z\end{pmatrix}-\frac{i}{2\lambda}[\sigma_3,\cdot]$$

and

$$Q(\partial,\bar\partial,\lambda)=-\frac{i}{2\lambda}[\sigma_3,(g-1)\cdot]+\frac{1-\sigma_3}{2}(g-1)\,\partial_z+\frac{1+\sigma_3}{2}(g-1)\,\partial_{\bar z}. \qquad (3.3)$$

The operator $\hat{G}$ is exactly the same as for the DS [14] and Ishimori [17] case and it acts as follows

$$(\hat{G}\Phi)(z,\bar z)=\iint\frac{dz'\wedge d\bar z'}{2\pi i}\times$$
$$\times\begin{vmatrix}\dfrac{\Phi_{11}(z',\bar z')}{z'-z}; & \dfrac{\Phi_{12}(z',\bar z')}{z'-z}\exp\left\{\dfrac{i(\bar z-\bar z')}{\lambda}+\dfrac{i(z-z')}{\bar\lambda}\right\}\\ \dfrac{\Phi_{21}(z',\bar z')}{\bar z'-\bar z}\exp\left\{-\dfrac{i(z-z')}{\lambda}-\dfrac{i(\bar z-\bar z')}{\bar\lambda}\right\}; & \dfrac{\Phi_{22}(z',\bar z')}{\bar z'-\bar z}\end{vmatrix} \qquad (3.4)$$

where $\Phi(z,\bar z)$ is an arbitrary 2×2 matrix. Note that the operator $\hat{G}$ is the bounded one.

It is easy to see from (3.4) that the Green function $G(\lambda,\bar\lambda)$ is nowhere analytic in λ. As a result, the solution of the equation (3.2) is nowhere analytic too.

Following the $\bar\partial$-method, we must construct now a corresponding $\bar\partial$-equation for χ. Differentiating equation (3.2) with respect to $\bar\lambda$ and taking account (3.4), we obtain

$$\frac{\partial\chi(z,\bar z,\lambda,\bar\lambda)}{\partial\bar\lambda}=$$
$$=\begin{vmatrix}0; & F_1(\lambda,\bar\lambda)\exp\left\{\dfrac{i\bar z}{\lambda}+\dfrac{iz}{\bar\lambda}\right\}\\ F_2(\lambda,\bar\lambda)\exp\left\{-\dfrac{iz}{\lambda}-\dfrac{i\bar z}{\bar\lambda}\right\}; & 0\end{vmatrix}-\hat{G}Q(\partial',\bar\partial',\lambda)\frac{\partial\chi}{\partial\bar\lambda}, \qquad (3.5)$$

where

$$F_1(\lambda,\bar\lambda)=\iint\frac{dz\wedge d\bar z}{\pi\bar\lambda^2}\exp\left\{-\frac{i\bar z}{\lambda}-\frac{iz}{\bar\lambda}\right\}[\,Q(\partial,\bar\partial,\lambda)\,(\chi(z,\bar z,\lambda)]_{12},$$

$$F_2(\lambda,\bar\lambda)=-\iint\frac{dz\wedge d\bar z}{\pi\bar\lambda^2}\exp\left\{\frac{iz}{\lambda}+\frac{i\bar z}{\bar\lambda}\right\}[\,Q(\partial,\bar\partial,\lambda)\,\chi(z,\bar z,\lambda)]_{21}, \qquad (3.6)$$

The terms proportional to $\delta(\lambda)$ which could appear in r.h.s. of (3.5) are equal to zero due to the special matrix structure of Q and the vanishing at $\lambda=0$ the integrals which contain the highly oscillating exponents similar to (3.6).

Then we introduce another solution $N(z,\bar z,\lambda,\bar\lambda)$ of equation (3.1) which obeys also the integral equation

$$N(z,\bar z,\lambda,\bar\lambda)=\Sigma_\lambda(z,\bar z)-(\hat{G}\,Q(\partial,\bar\partial,\lambda)\,N(\ldots,\lambda))\,(z,\bar z), \qquad (3.7)$$

where

$$\Sigma_\lambda(z,\bar z)=\begin{vmatrix}0 & ; & \exp\left\{\dfrac{i\bar z}{\lambda}+\dfrac{iz}{\bar\lambda}\right\}\\ \exp\left\{-\dfrac{iz}{\lambda}-\dfrac{i\bar z}{\bar\lambda}\right\} & ; & 0\end{vmatrix}. \qquad (3.8)$$

Comparing equations (3.5) and (3.7) and assuming that the homogeneous equation (3.7) has no nontrivial solutions, one obtains

$$\frac{\partial \chi(z,\bar{z},\lambda,\bar{\lambda})}{\partial \bar{\lambda}} = N(z,\bar{z},\lambda,\bar{\lambda}) \begin{pmatrix} F_2, & 0 \\ 0, & F_1 \end{pmatrix}. \tag{3.9}$$

Now it is necessary to establish the relation between functions N and χ. Using the integral equation (3.2) and (3.7) and the identity

$$[\,G(\dots,\lambda)\, Q(\dots,\lambda)\, \Phi \Sigma_\lambda] = [\,G(\dots,\bar{\lambda})\, Q(\dots,\bar{\lambda})\, \Phi]\, \Sigma_\lambda \tag{3.10}$$

one finds

$$N(z,\bar{z},\lambda,\bar{\lambda}) = \chi\,(z,\bar{z},\bar{\lambda},\lambda)\, \Sigma_\lambda(z,\bar{z})\,. \tag{3.11}$$

Substituting the expression (3.11) into (3.9), we finally arrive at the linear $\bar{\partial}$-equation

$$\frac{\partial}{\partial \bar{\lambda}} \chi\,(z,\bar{z},\lambda,\bar{\lambda}) = \chi\,(z,\bar{z},\bar{\lambda},\lambda)\, F(\lambda,\bar{\lambda},z,\bar{z})\,, \tag{3.12}$$

where

$$F(\lambda,\bar{\lambda},z,\bar{z}) = \begin{vmatrix} 0\,; & F_1(\lambda,\bar{\lambda}) \exp\left\{\frac{i\bar{z}}{\lambda} + \frac{iz}{\bar{\lambda}}\right\} \\ F_2(\lambda,\bar{\lambda}) \exp\left\{-\frac{iz}{\lambda} - \frac{i\bar{z}}{\bar{\lambda}}\right\}; & 0 \end{vmatrix} \tag{3.13}$$

and F_1 and F_2 are given by (3.6).

In order to complete equation (3.12) one should add also the information about the singular points of the function χ. We will assume that the homogeneous equation (3.2) has a finite number of simple points $\lambda_1, \dots, \lambda_n$. This implies that the solutions of equation (3.2) have a form

$$\chi(z,\bar{z},\lambda,\bar{\lambda}) = \sum \frac{\Phi_i(z,\bar{z})}{\lambda - \lambda_i} + \tilde{\chi}(z,\bar{z},\lambda,\bar{\lambda})\,, \tag{3.14}$$

where Φ_i are the solutions of the homogeneous equation (3.2) and $\tilde{\chi}$ is a function bounded in λ. A precise structure of the singular part S of the function χ can be determined by the use of the following two properties of equation (3.2). Firstly, each column of the 2×2 matrix Φ_i obeys the homogeneous equation (3.2) separately. Therefore, the columns $\begin{pmatrix}\Phi^i_{11}\\ \Phi^i_{21}\end{pmatrix}$ and $\begin{pmatrix}\Phi^i_{12}\\ \Phi^i_{22}\end{pmatrix}$ can be the solution of the homogeneous equation (2.4) in different points λ_i and μ_k. Secondly,

it follows from the identity (3.10) that if the matrix $\begin{pmatrix}\Phi_{11}, & 0\\ \Phi_{21}, & 0\end{pmatrix}$ is the solution of the homogeneous equation (3.2) at the point λ_i then the matrix

$$\begin{pmatrix}0, & \Phi_{11}\\ 0, & \Phi_{21}\end{pmatrix} \exp\left(\frac{i\bar{z}}{\bar{\lambda}} + \frac{iz}{\lambda}\right)$$

is the solution of the homogeneous equation (3.2) at the point $\bar{\lambda}_i$. As a consequence of these two facts the singular part S of the function χ is if the form

$$S_{\alpha 1} = \sum_{k=1}^{n1} \frac{\Phi^{(k)}_{\alpha 1}}{\lambda - \lambda_k} + \sum_{l=1}^{n2} \frac{\Phi^{(l)}_{\alpha 2}}{\lambda - \bar{\mu}_l} \exp\left\{-\frac{i\bar{z}}{\mu_l} - \frac{iz}{\bar{\mu}_l}\right\},$$

$$S_{\alpha 2} = \sum_{l=1}^{n2} \frac{\Phi^{(l)}_{\alpha 2}}{\lambda - \mu_l} + \sum_{k=1}^{n1} \frac{\Phi^{(k)}_{\alpha 1}}{\lambda - \bar{\lambda}_k} \exp\left\{\frac{i\bar{z}}{\bar{\lambda}_k} + \frac{iz}{\lambda_k}\right\}, \quad (\alpha = 1, 2)\,. \tag{3.15}$$

Linear $\bar{\partial}$-equation plays a fundamental role in the IST method: it generates the equation of the inverse problem for the spectral problem (3.1). Indeed, taking into account (2.18) and using the generalized Cauchy formula (see, e. g. [19])

$$\chi(\lambda) = \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i} \frac{\frac{\partial \chi(\lambda')}{\partial \bar{\lambda}'}}{\lambda' - \lambda} + \int_{C_\infty} \frac{d\lambda'}{2\pi i} \frac{\chi(\lambda')}{\lambda' - \lambda}, \tag{3.16}$$

where C is the entire complex plane, we obtain from (3.12) the following integral equation

$$\chi(z,\bar{z},\lambda,\bar{\lambda}) = 1 + S(z,\bar{z},\lambda,\bar{\lambda}) + \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i} \frac{\chi(z,\bar{z},\bar{\lambda}',\lambda')\, F(\lambda',\bar{\lambda}')}{\lambda' - \lambda}, \tag{3.17}$$

where S is given by (3.15).

The two-dimensional singular integral equation (3.17) is the basic equation of the inverse problem for the spectral problem (3.1). In order to extract the complete set of the inverse problem equations from (3.17) one should use also the relation

$$\lim_{\lambda \to \lambda_i} \left(\chi - \frac{\Phi^{(i)}}{\lambda - \lambda_i}\right) = \Phi^{(i)} \begin{pmatrix} iz/\lambda_i^2 + \gamma_{1i}, & 0 \\ 0, & -i\bar{z}/\lambda_i^2 + \gamma_{2i} \end{pmatrix}, \tag{3.18}$$

where γ_i are some constants. The relation can be proved similar to the DS-I and Ishimori-I equations cases [12, 13, 17].

Taking into account (3.18) and proceeding in (3.17) to the limits $\lambda \to \lambda_i$, $\lambda \to \mu_k$, one gets the system of equations

$$\delta_{\alpha 1} - \Phi_{\alpha 1}^{(i)} \left(\frac{iz}{\lambda_i^2} + \gamma_{1i} \right) + \sum_{k \neq i}^{n_1} \frac{\Phi_{\alpha 1}^{(k)}}{\lambda_i - \lambda_k} + \sum_{l=1}^{n_2} \frac{\Phi_{\alpha 2}^{(l)}}{\lambda_i - \bar{\mu}_l} \exp\left\{ -\frac{i\bar{z}}{\mu_l} - \frac{iz}{\bar{\mu}_l} \right\} +$$

$$+ \iint \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i} \frac{\chi(z, \bar{z}, \bar{\lambda}', \lambda') F(\lambda', \bar{\lambda}', z, \bar{z})}{\lambda' - \lambda_i} = 0 \quad (\alpha = 1, 2; \quad i = 1, \ldots, n_1),$$

$$\delta_{\alpha 2} - \Phi_{\alpha 2}^{(j)} \left(-\frac{i\bar{z}}{\mu_j^2} + \gamma_{2j} \right) + \sum_{l \neq j}^{n_2} \frac{\Phi_{\alpha 2}^{(l)}}{\mu_j - \mu_l} + \sum_{k=1}^{n_1} \frac{\Phi_{\alpha 1}^{(k)}}{\mu_j - \bar{\lambda}_k} \exp\left\{ \frac{i\bar{z}}{\bar{\lambda}_k} + \frac{iz}{\lambda_k} \right\} +$$

$$+ \iint \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i} \frac{\chi(z, \bar{z}, \bar{\lambda}', \lambda') F(\lambda', \bar{\lambda}', z, \bar{z})}{\lambda' - \mu_j} = 0 \quad (\alpha = 1, 2; \quad j = 1, \ldots, n_2). \tag{3.19}$$

At last, directly from the problem (3.1) (see section 2.3) one has

$$g(z, \bar{z}, t) = \chi_0^{-1}; \qquad C(z, \bar{z}, t) = -(\chi_0^{-1} \chi_1)_{diag}, \tag{3.20}$$

where

$$\chi_0 \stackrel{def}{=} \chi(z, \bar{z}, \lambda, \bar{\lambda})|_{\lambda=0} \quad \text{and} \quad \chi_1(z, \bar{z}, t) \stackrel{def}{=} \frac{\partial}{\partial \lambda} \chi(z, \bar{z}, \lambda, \bar{\lambda})|_{\lambda=0}.$$

Equation (3.17), (3.19), and formulae (3.20) form the complete set equations which solve the inverse problem for the spectral problem (3.1). The set

$$\mathscr{F}(\lambda, \bar{\lambda}) = \{F_1(\lambda, \bar{\lambda}) \,;\, F_2(\lambda, \bar{\lambda}) \,;\, \lambda_i, \gamma_{1i} (i = 1, \ldots, n_1) \,;\, \mu_k, \gamma_{2k} (k = 1, \ldots, n_2)\}$$

is the inverse problem data for the problem (3.1). Given $\mathscr{F}(\lambda, \bar{\lambda})$, one can calculate functions χ, $\Phi^{(i)}$ with the use of the integral equations (3.17), (3.19). Finally, one reconstructs the variables $g(x, y, t)$ and $C(x, y, t)$ by the formulae (3.20) where

$$\chi_0 = 1 + S|_{\lambda=0} + \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i} \frac{\chi(z, \bar{z}, \bar{\lambda}', \lambda') F(\lambda', \bar{\lambda}', z, \bar{z})}{\lambda'},$$

$$\chi_1 = \frac{\partial S}{\partial \lambda}\Big|_{\lambda=0} + \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i} \frac{\chi(z, \bar{z}, \bar{\lambda}', \lambda') F(\lambda', \bar{\lambda}', z, \bar{z})}{\lambda'^2}, \tag{3.21}$$

Note that the equations (3.17), (3.19) are solvable at least for the small data $F(\lambda, \bar{\lambda})$.

Now we are able to solve the initial value problem for equation (1.1). For this purpose one must establish the evolution of the inverse problem data $\mathscr{F}(\lambda, \bar{\lambda})$ in time t. Using equation (2.14b), one, in a standard manner, gets

$$\frac{dF_1(\lambda, \bar{\lambda}, t)}{dt} = \frac{i}{2} \left(\frac{1}{\lambda^2} + \frac{1}{\bar{\lambda}^2} \right) F_1(\lambda, \bar{\lambda}, t),$$

$$\frac{dF_2(\lambda, \bar{\lambda}, t)}{dt} = -\frac{i}{2} \left(\frac{1}{\lambda^2} + \frac{1}{\bar{\lambda}^2} \right) F_2(\lambda, \bar{\lambda}, t),$$

$$\frac{d\lambda_k}{dt} = \frac{d\mu_k}{dt} = 0,$$

$$\frac{d\gamma_{1i}}{dt} = \frac{i}{\lambda_i^3}, \qquad \frac{d\gamma_{2i}}{dt} = -\frac{i}{\mu_i^3}. \tag{3.22}$$

Therefore

$$F_1(\lambda, \bar{\lambda}, t) = F_1(\lambda, \bar{\lambda}, 0) \exp\left[\frac{i}{2} \left(\frac{1}{\lambda^2} + \frac{1}{\bar{\lambda}^2} \right) t \right],$$

$$F_2(\lambda, \bar{\lambda}, t) = F_2(\lambda, \bar{\lambda}, 0) \exp\left[-\frac{i}{2} \left(\frac{1}{\lambda^2} + \frac{1}{\bar{\lambda}^2} \right) t \right], \tag{3.23}$$

$$\gamma_{1i} = \gamma_{1i_0} + \frac{i}{\lambda_i^3} t, \qquad \gamma_{2i}(t) = \gamma_{2i_0} - \frac{i}{\mu_i^3} t,$$

where γ_{1i_0} and γ_{2i_0} are arbitrary constants.

The use of the formula (3.23) allows us to solve the initial value problem for equation (1.1) by the IST method standard procedure

$$g(x, y, t) \xrightarrow{(3.16)} \mathscr{F}(\lambda, \bar{\lambda}, 0) \xrightarrow{(3.23)} \mathscr{F}(\lambda, \bar{\lambda}, t) \to$$

$$\xrightarrow{(3.17), (3.19), (3.20)} g(x, y, t). \tag{3.24}$$

Emphasize that the procedure described gives the solution of the initial value problem for the variable g which tends to the asymptotic value (unit) sufficiently fastly as $x^2 + y^2 \to \infty$.

As usual (see e. g. [10—13, 17]), one can find the solution of equation (1.1) which correspond to the case $F_{1,2}(\lambda, \bar{\lambda}, t) = 0$ in an explicit form. Indeed, in this case the system (3.19) is the linear algebraic system which can be easily solved with respect to $\Phi_{\alpha\beta}^{(i)}$. Substituting these $\Phi_{\alpha\beta}^{(i)}$ into (3.17), we obtain the solutions of equation (1.1) given by the formula (3.20) where $\chi_0 = 1 + S|_{\lambda=0}$ and

$\chi_1=\frac{\partial}{\partial\lambda}S|_{\lambda=0}$. The simplest solution corresponds to the case $n_1=n_2=1$ and it is

$$\chi_0=1-\frac{1}{\pi}\times$$

$$\times\begin{vmatrix} \frac{-i\bar z/\mu^2+it/\mu^3+\gamma_2}{\lambda}+\frac{\lambda-\bar\mu}{\bar\mu}e^+e^-; & \left(\frac{-i\bar z/\mu^2+it/\mu^3+\gamma_2}{\bar\lambda}(\mu-\bar\lambda)+\frac{1}{\mu}\right)e^+ \\ \left(\frac{1}{\lambda}+\frac{\lambda-\bar\mu}{\bar\mu}\left(\frac{iz}{\lambda^2}-\frac{it}{\lambda^3}+\gamma_1\right)\right)e^-; & \frac{iz/\lambda^2-it/\lambda^3+\gamma_1}{\mu}+\frac{\mu-\bar\lambda}{\bar\lambda}e^+e^- \end{vmatrix} \tag{3.25}$$

and

$$\chi_1=-\frac{1}{\pi}\times$$

$$\times\begin{vmatrix} \frac{-i\bar z/\mu^2+it/\mu^3+\gamma_2}{\lambda^2}+\frac{\lambda-\bar\mu}{\bar\mu^2}e^+e^-; & \left(\frac{-i\bar z/\mu^2+it/\mu^3+\gamma_2}{\bar\lambda^2}(\mu-\bar\lambda)+\frac{1}{\mu^2}\right)e^+ \\ \left(\frac{1}{\lambda^2}+\frac{\lambda-\bar\mu}{\bar\mu^2}\left(\frac{iz}{\lambda^2}-\frac{it}{\lambda^3}+\gamma_1\right)\right)e^-; & \frac{iz/\lambda^2-it/\lambda^3+\gamma_1}{\mu^2}+\frac{\mu-\bar\lambda}{\bar\lambda^2}e^+e^- \end{vmatrix} \tag{3.26}$$

where

$$\pi=\left(\frac{iz}{\lambda^2}-\frac{it}{\lambda^3}+\gamma_1\right)\left(-\frac{i\bar z}{\mu^2}+\frac{it}{\mu^3}+\gamma_2\right)-e^+e^-,$$

$$e^+=(\mu-\bar\lambda)^{-1}\exp\left\{\frac{i\bar z}{\bar\lambda}+\frac{iz}{\lambda}\right\}, \tag{3.27}$$

$$e^-=(\lambda-\bar\mu)^{-1}\exp\left\{-\frac{i\bar z}{\mu}-\frac{iz}{\bar\mu}\right\}$$

The formulae (2.29), (2.30) and (2.35), (2.36) give us the corresponding rational-exponential solutions of the DS and Ishimori equations, found in [13] and [17].

4. INITIAL VALUE PROBLEM AT $\alpha=1$

In the case $\alpha=1$ the linear problem (1.2a) is the hyperbolic linear system. The corresponding equation (2.14a) for the function χ is $\left(\xi=\frac{1}{2}(y+x),\ \eta=\frac{1}{2}(y-x)\right)$

$$\begin{pmatrix}\partial_\eta & 0\\ 0 & \partial_\xi\end{pmatrix}\chi-\frac{i}{2\lambda}[\sigma_3,\chi]-\frac{i}{2\lambda}[\sigma_3,(g-1)\chi]+$$

$$+\frac{1-\sigma_3}{2}(g-1)\,\partial_\xi\chi+\frac{1+\sigma_3}{2}(g-1)\,\partial_\eta\chi=0. \tag{4.1}$$

The operator $\hat G$ inverse to

$$L_0=\begin{pmatrix}\partial_\eta & 0\\ 0 & \partial_\xi\end{pmatrix}-\frac{i}{2\lambda}[\sigma_3,\cdot]$$

can be calculated by the same method as in the previous section. It is of the form (for Ishimori equation see [17])

$$(\hat G\Phi)(\xi,\eta)\overset{def}{=}\begin{vmatrix} \partial_\eta^{-1}(\Phi_{11}(\xi,\eta')); & \partial_\eta^{-1}\left(\exp\left\{\frac{i(\eta-\eta')}{\lambda}\right\}\Phi_{12}(\xi,\eta')\right) \\ \partial_\xi^{-1}\left(\exp\left\{-\frac{i(\xi-\xi')}{\lambda}\right\}\Phi_{21}(\xi',\eta)\right); & \partial_\xi^{-1}(\Phi_{22}(\xi',\eta)) \end{vmatrix} \tag{4.2}$$

where Φ is an arbitrary 2×2 matrix.

The main feature of the operator (4.2), in comparison with (3.4), consists in the absence of $\bar\lambda$-dependence. Thus, the operator (4.2) is analytic function on the entire complex plane of λ.

Another feature of the operator (4.2) is that it is defined non-uniquelly. The freedom in the definition of this operator is connected with the possibility to choose the different concrete realizations of the formal operators ∂_η^{-1} and ∂_ξ^{-1}. This freedom can be used for the construction of the bounded operators $\hat G$. Indeed, choosing

$$\partial_\eta^{-1}f\overset{def}{=}\int_{+\infty}^{\eta}d\eta' f(\eta')\quad\text{and}\quad\partial_\xi^{-1}f\overset{def}{=}\int_{-\infty}^{\xi}d\xi' f(\xi'),$$

we define the operator $\hat G^+$:

$$(\hat G^+(\cdot,\lambda)\,\Phi(\cdot))(\xi,\eta)=$$

$$=\begin{vmatrix} \int_{+\infty}^{\eta}d\eta'\Phi_{11}(\xi,\eta'); & \int_{+\infty}^{\eta}d\eta'\exp\left\{\frac{i(\eta-\eta')}{\lambda}\right\}\Phi_{12}(\xi,\eta') \\ \int_{-\infty}^{\xi}d\xi'\exp\left\{-\frac{i(\xi-\xi')}{\lambda}\right\}\Phi_{21}(\xi',\eta); & \int_{-\infty}^{\xi}d\xi'\Phi_{22}(\xi',\eta) \end{vmatrix} \tag{4.3}$$

The choice

$$\partial_\xi^{-1}f\overset{def}{=}\int_{+\infty}^{\xi}d\xi' f(\xi',\eta)\quad\text{and}\quad\partial_\eta^{-1}f\overset{def}{=}\int_{-\infty}^{\eta}d\eta' f(\xi,\eta')$$

gives the operator $\hat G^-$:

$$(\hat{G}^-(\cdot,\lambda)\,\Phi(\cdot))(\xi,\eta)=$$

$$=\left|\begin{matrix} \int_{-\infty}^{\eta} d\eta'\Phi_{11}(\xi,\eta'); & \int_{-\infty}^{\eta} d\eta \exp\left\{\frac{i(\eta-\eta')}{\lambda}\right\}\Phi_{12}(\xi,\eta') \\ \int_{+\infty}^{\xi} d\xi'\exp\left\{-\frac{i(\xi-\xi')}{\lambda}\right\}\Phi_{21}(\xi',\eta); & \int_{+\infty}^{\xi} d\xi'\Phi_{22}(\xi',\eta)\end{matrix}\right| \qquad (4.4)$$

It is easy to see that the operator $\hat{G}^+(\lambda)$ is bounded at the upper half-plane $\mathrm{Im}\,\lambda>0$, while the operator $\hat{G}^-(\lambda)$ is bounded at lower half-plane.

Now let us introduce the solutions χ^+ and χ^- of the problem (4.1) which simultaneously are the solutions of the integral equations

$$\chi^{\pm}(\xi,\eta,\lambda)=1-(\hat{G}^{\pm}(\cdot,\lambda)\,Q(\partial,\bar\partial,\lambda)\,\chi^{\pm}(\cdot,\lambda))(\xi,\eta), \qquad (4.5)$$

where G^+ and G^- are given by the formulae (4.3), (4.4) and

$$Q \overset{def}{=} -\frac{i}{2\lambda}|\sigma_3,(g-1)\cdot|+\frac{1-\sigma_3}{2}(g-1)\,\partial_\xi+\frac{1-\sigma_3}{2}(g-1)\,\partial_\eta .$$

As far as the Green functions G^+ and G^-, the solutions χ^+ and χ^- are analytic and bounded functions in the upper and lower half-planes respectively. Further, since $G^+-G^-\neq 0$ at $\mathrm{Im}\,\lambda=0$, then $\chi^+-\chi^-\neq 0$ at $\mathrm{Im}\,\lambda=0$ too. Thus, one can define the function

$$\chi(\xi,\eta,\lambda)=\begin{cases}\chi^+, & \mathrm{Im}\,\lambda>0\\ \chi^-, & \mathrm{Im}\,\lambda<0\end{cases}$$

which is analytic and bounded on the entire complex plane and has a jump across the real axis. So, we arrive at the standard Riemann—Hilbert problem. We will also assume that the homogeneous equation (4.5) has no nontrivial solutions.

At this stage, according to the standard procedure (see [19]), one must find out the relation between the functions χ^+ and χ^- on the real axis. This relation is similar to that for the Ishimori equation [17].

Firstly, we note that equations (4.5) straightforwardly give

$$(\chi^+-\chi^-)(\xi,\eta,\lambda)=\Gamma(\xi,\eta,\lambda)-[\,\hat{\hat{G}}(\cdot,\lambda)\,Q(\partial,\bar\partial,\lambda)\,(\chi^+-\chi^-)]\,(\xi,\eta), \qquad (4.6)$$

where

$$(\hat{\hat{G}}(\cdot,\lambda)\,\Phi)\,(\xi,\eta)=$$

$$=\left|\begin{matrix} \int_{-\infty}^{\eta} d\eta'\Phi_{11}(\xi,\eta'); & \int_{-\infty}^{\eta} d\eta'\exp\left\{\frac{i(\eta-\eta')}{\lambda}\right\}\Phi_{12}(\xi,\eta') \\ \int_{-\infty}^{\xi} d\xi'\exp\left\{-\frac{i(\xi-\xi')}{\lambda}\right\}\Phi_{21}(\xi',\eta); & \int_{-\infty}^{\xi} d\xi'\Phi_{22}(\xi',\eta)\end{matrix}\right| \qquad (4.7)$$

and

$$\Gamma(\xi,\eta,\lambda)=$$

$$=\left|\begin{matrix} -\int_{-\infty}^{+\infty} d\eta'\,[Q(\partial,\bar\partial,\lambda)\,\chi^+(\xi,\eta')]_{11}; & -\int_{-\infty}^{+\infty} d\eta'\exp\left\{\frac{i(\eta-\eta')}{\lambda}\right\}[Q(\partial,\bar\partial,\lambda)\chi^+]_{12} \\ -\int_{-\infty}^{+\infty} d\xi'\exp\left\{-\frac{i(\xi-\xi')}{\lambda}\right\}[Q(\partial,\bar\partial,\lambda)\,\chi^-]_{21}; & \int_{-\infty}^{+\infty} d\xi'\,[Q(\partial,\bar\partial,\lambda)\,\chi^-]_{22}\end{matrix}\right| \qquad (4.8)$$

Then we introduce the 2×2 matrix $f(l,k)$ defined by the relation

$$f(l,k)-\int_{-\infty}^{+\infty} dk'\,T^-(l,k')\,f(k',k)=T^-(l,k)-T^+(l,k), \qquad (4.9)$$

where

$$T_{21}^-(l,k)=\iint\frac{d\xi\,d\eta}{2\pi}\exp\left\{\frac{i\eta}{l}-\frac{i\xi}{k}\right\}[\,Q(\partial,\bar\partial,k)\,\chi^-(\xi,\eta,k)]_{21},$$

$$T_{11}^-=T_{12}^-=T_{22}^-=0 \qquad (4.10)$$

and

$$T_{12}^+(l,k)=\iint\frac{d\xi d\eta}{2\pi}\exp\left\{-\frac{i\xi}{l}-\frac{i\eta}{k}\right\}[\,Q(\partial,\bar\partial,k)\,\chi^+(\xi,\eta,k)]_{12},$$

$$T_{11}^+=T_{22}^+=T_{21}^+=0. \qquad (4.11)$$

The integral equation (4.9), in fact, is easily solved and one gets

$$f(l,k)=\left|\begin{matrix}0, & T_{12}^+(l,k)\\ T_{21}^-(l,k), & -\int_{-\infty}^{+\infty} dk'\,T_{21}^-(l,k')\,T_{12}^+(k',k)\end{matrix}\right| \qquad (4.12)$$

Further, the obvious identity

$$Q(\partial,\bar\partial,k)\,\chi^-(\xi,\eta,k)\,\Sigma_l(\xi,\eta)\,f(l,k)\,\Sigma_k^{-1}(\xi,\eta)=$$

$$=[\,Q(\partial,\bar\partial,l)\,\chi^-(\xi,\eta,k)]\,\Sigma_l(\xi,\eta)\,f(l,k)\,\Sigma_k^{-1}(\xi,\eta) \qquad (4.13)$$

holds, where

$$\Sigma_\lambda(\xi,\eta) \stackrel{def}{=} \begin{pmatrix} e^{i\xi/\lambda} & 0 \\ 0 & e^{-i\eta/\lambda} \end{pmatrix}. \tag{4.14}$$

At last, using (4.13), one can straightforwardly show that the quantity

$$\int_{-\infty}^{+\infty} dk\,\chi^-(\xi,\eta,k)\,\Sigma_k(\xi,\eta)\,f(k,\lambda)\,\Sigma_\lambda^{-1}(\xi,\eta)$$

obeys the same equation (4.6) as $(\chi^+ - \chi^-)(\xi,\eta,\lambda)$. In virtue of the absence of the nontrivial solutions for the homogeneous equation (4.6) this gives

$$\chi^+(\xi,\eta,\lambda) - \chi^-(\xi,\eta,\lambda) = \int_{-\infty}^{+\infty} dl\,\chi^-(\xi,\eta,l)\,\Sigma_l(\xi,\eta)\,f(l,\lambda)\,\Sigma_\lambda^{-1}(\xi,\eta). \tag{4.15}$$

Thus, the jump $\chi^+ - \chi^-$ at $\operatorname{Im}\lambda = 0$ is expressed linearly and nonlocally via χ^-. So, we have the regular nonlocal Riemann—Hilbert problem.

With the use of the standard formulae which solve the standard Riemann—Hilbert problem (see e. g. [7, 19]) we obtain from (4.15) the following integral equation

$$\chi^-(\xi,\eta,\lambda) - \iint_{-\infty}^{+\infty} \frac{dl\,dk}{2\pi i}\,\frac{\chi^-(\xi,\eta,l)\,\Sigma_l(\xi,\eta)\,f(l,k)\,\Sigma_k^{-1}(\xi,\eta)}{k-\lambda+i0} = 1. \tag{4.16}$$

Further, directly from (4.1) similar to (3.20) one has

$$g = \chi_0^{-1}, \qquad C = i(\chi_0^{-1}\chi_1)_{diag}. \tag{4.17}$$

From equation (4.16) it follows that

$$\chi_0 = 1 + \iint_{-\infty}^{+\infty} \frac{dl\,dk}{2\pi i}\,\frac{\chi^-(\xi,\eta,l)\,\Sigma_l(\xi,\eta)\,f(l,k)\,\Sigma_k^{-1}(\xi,\eta)}{k},$$

$$\chi_1 = \iint_{-\infty}^{+\infty} \frac{dl\,dk}{2\pi i}\,\frac{\chi^-(\xi,\eta,l)\,\Sigma_l(\xi,\eta)\,f(l,k)\,\Sigma_k^{-1}(\xi,\eta)}{k^2}. \tag{4.18}$$

The integral equation (4.16) and formulae (4.17), (4.18) are the

complete set of the inverse problem equations for the spectral problem (4.1). The functions $T_{21}^-(l,k)$ and $T_{12}^+(l,k)$ are the inverse problem data.

To solve the initial value problem for equation (1.1) with $\alpha = 1$ one should to find, as usual, the time evolution of the inverse problem data. The evolution laws of the functions T_{12}^+ and T_{21}^- can be found in a standard manner from the second auxiliary linear problem (2.14b) with $\alpha = 1$. One obtains

$$T_{12}^+(k,\lambda,t) = T_{12}^+(k,\lambda,0)\exp\left\{\frac{1}{2}\left(\frac{1}{k^2}+\frac{1}{\lambda^2}\right)t\right\},$$

$$T_{21}^-(k,\lambda,t) = T_{21}^-(k,\lambda,0)\exp\left\{-\frac{1}{2}\left(\frac{1}{k^2}+\frac{1}{\lambda^2}\right)t\right\}. \tag{4.19}$$

The inverse problem equations (4.16) — (4.18) and the evolution law (4.19) allow us to solve, in principle, the initial value problem for equation (1.1) with $\alpha = 1$ by the standard procedure (3.24).

The formulae (4.16) — (4.18) give the possibility to construct exact solutions of equation (1.1) with $\alpha = 1$. In particular, one can construct exact solutions with the functional parameters which are typical for equations connected with the nonlocal Riemann—Hilbert problem [22]. The solutions of such type correspond to the factorized functions T_{12}^+ and T_{21}^- of the form

$$T_{12}^+(l,k,t) = \sum_{n=1}^{N} f_n^+(l)\,g_n^+(k)\exp\left\{\frac{i}{2}\left(\frac{1}{l^2}+\frac{1}{k^2}\right)t\right\},$$

$$T_{21}^-(l,k,t) = \sum_{n=1}^{N} f_n^-(l)\,g_n^-(k)\exp\left\{-\frac{i}{2}\left(\frac{1}{l^2}+\frac{1}{k^2}\right)t\right\}, \tag{4.20}$$

where $f_n^\pm(l)$ and $g_n^\pm(k)$ are arbitrary functions. The solution induced by (4.20) depends on $4N$ arbitrary functions of the single variable. The expression for χ_0 has been given, in fact, in the paper [17].

In the next section we will construct more general solutions with functional parameters both for equation (1.1) with $\alpha = i$ and $\alpha = 1$.

5. EXACT SOLUTIONS VIA $\bar{\partial}$-DRESSING METHOD

In this section we will consider equation (1.1) within the framework of the $\bar{\partial}$-dressing method proposed in [22—24] (see also a review in [21]). The $\bar{\partial}$-dressing method is the generalization of the original Zakharov—Shabat dressing method (see [1, 2]). The aim of the dressing method is to construct the nonlinear integrable equations and simultaneously their exact solutions.

A starting point of the $\bar{\partial}$-dressing method is the nonlocal $\bar{\partial}$-problem [22]

$$\frac{\partial\chi(\lambda,\bar{\lambda})}{\partial\bar{\lambda}}=(\chi*R)(\lambda,\bar{\lambda})=\iint_C d\lambda'\wedge d\bar{\lambda}'\chi(\lambda',\bar{\lambda}')R(\lambda',\bar{\lambda}',\lambda,\bar{\lambda}), \tag{5.1}$$

where χ and R are matrix-valued functions. We will assume that the function χ is canonically normalized $\chi\underset{\lambda\to\infty}{\longrightarrow}1$ and the $\bar{\partial}$-equation (1.1) is solved uniquelly. A dependence on the variables x, y and t is introduced by the following dependence of R on x, y, t:

$$\frac{\partial R}{\partial x}=\frac{i\alpha}{2}\left(-\frac{i}{\lambda'}R(\lambda',\bar{\lambda}',\lambda,\bar{\lambda},x,y,t)+R(\lambda',\bar{\lambda}',\lambda,\bar{\lambda},x,y,t)\frac{1}{\lambda}\right),$$

$$\frac{\partial R}{\partial y}=\frac{i}{2}\left(-\frac{\sigma_3}{\lambda'}R(\lambda',\bar{\lambda}',\lambda,\bar{\lambda},x,y,t)+R(\lambda',\bar{\lambda}',\lambda,\bar{\lambda},x,y,t)\frac{\sigma_3}{\lambda}\right),$$

$$\frac{\partial R}{\partial t}=\frac{i\alpha^2}{2}\left(-\frac{\sigma_3}{\lambda'^2}R(\lambda',\bar{\lambda}',\lambda,\bar{\lambda},x,y,t)+R(\lambda',\bar{\lambda}',\lambda,\bar{\lambda},x,y,t)\frac{\sigma_3}{\lambda^2}\right), \tag{5.2}$$

i. e.

$$R(\lambda',\bar{\lambda}',\lambda,\bar{\lambda},x,y,t)=\exp\left\{-\frac{i}{2}\left(\frac{\alpha}{\lambda'}x+\frac{\sigma_3}{\lambda'}y+\alpha^2\frac{\sigma_3}{\lambda'^2}t\right)\right\}\times$$

$$\times R(\lambda',\bar{\lambda}',\lambda,\bar{\lambda},0,0,0)\exp\left\{\frac{i}{2}\left(\frac{\alpha}{\lambda}x+\frac{\sigma_3}{\lambda}y+\alpha^2\frac{\sigma_3}{\lambda^2}t\right)\right\}, \tag{5.3}$$

where $R(\lambda', \bar{\lambda}', \lambda, \bar{\lambda}, 0, 0, 0)$ is an arbitrary matrix-valued function. Then the «long» derivatives D_x, D_y and D_t are introduced

$$D_xf\overset{def}{=}\partial_xf+\frac{i\alpha}{2\lambda}f,$$

$$D_yf\overset{def}{=}\partial_yf+\frac{i}{2\lambda}f\sigma_3, \tag{5.4}$$

$$D_tf\overset{def}{=}\partial_tf+\frac{i\alpha^2}{2\lambda^2}f\sigma_3.$$

Further, the main intermediate purpose is to construct the operators L of the form

$$L=\sum_{n,m,l}u_{nml}(x,y,t)\,D_x^nD_y^mD_t^l, \tag{5.5}$$

which have no singularities on λ, i. e. which obey the condition [23]

$$\left[\frac{\partial}{\partial\bar{\lambda}},L\right]=0. \tag{5.6}$$

For such operators L_i the functions $L_i\chi$ obey the same $\bar{\partial}$-equation as a function χ and in virtue of the proposed unique solvability of equation (5.1) one has [23]

$$L_i\chi=0. \tag{5.7}$$

The system (5.7) is just the linear system which generates the integrable equation [22, 23].

In our case one can construct, as it is not difficult to show, the two operators L_1 and L_2 which obey the condition (5.6). For instance,

$$L_1=\alpha gD_y-\sigma_3gD_x,$$

$$L_2=igD_t-2\alpha gD_xD_y-2\alpha g_xD_y-2\alpha g_yD_x, \tag{5.8}$$

where

$$g(x,y,t)=(\chi(\lambda,\bar{\lambda},x,y,t)\,|_{\lambda=0})^{-1}. \tag{5.9}$$

The corresponding linear system (5.7) is nothing but the system (2.14). This linear system give rises, as it was shown in the section 2, to the integrable equation (1.1).

The formula (5.9) together with the formula

$$C(x,y,t)=i\alpha\left(g\frac{\partial}{\partial\lambda}\chi(\lambda,\bar{\lambda},x,y,t)\,|_{\lambda=0}\right)_{diag} \tag{5.10}$$

are the dressing formulae which give the solution of equation (1.1). Indeed, starting with given matrix $R(\lambda', \bar{\lambda}', \lambda, \bar{\lambda}, 0, 0, 0)$, one finds the corresponding solution χ of the $\bar{\partial}$-equation (5.1) which is equivalently, the solution of the integral equation

$$\chi(\lambda,\bar{\lambda})=1+\iint_C\frac{d\lambda'\wedge d\bar{\lambda}'}{2\pi i}\,\frac{(\chi*R)(\lambda',\bar{\lambda}')}{\lambda'-\lambda}, \tag{5.11}$$

where $R(\lambda', \bar{\lambda}', \lambda, \bar{\lambda}, x, y, t)$ is given by (5.3). Then the formulae (5.9) and (5.10) where

$$\chi(\lambda, \bar{\lambda})|_{\lambda=0} = 1 + \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i} \frac{(\chi * R)(\lambda', \bar{\lambda}')}{\lambda'}, \tag{5.12}$$

$$\frac{\partial}{\partial\lambda}\chi(\lambda, \bar{\lambda})\Big|_{\lambda=0} = \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i} \frac{(\chi * R)(\lambda', \bar{\lambda}')}{(\lambda')^2}, \tag{5.13}$$

give the solution of equation (1.1).

Emphasize that within the $\bar{\partial}$-dressing method it is assumed nothing about the behaviour of g at $x, y \to \infty$.

The formulae (5.9) — (5.13) allow as to construct the wide classes of the exact solutions of equation (1.1).

One of the interesting class of exact solutions, namely, the solutions with functional parameters correspond to the factorized functions $R(\lambda', \bar{\lambda}', \lambda, \bar{\lambda}, 0, 0, 0)$:

$$R(\lambda', \bar{\lambda}', \lambda, \bar{\lambda}, x, y, t) = e^{F(\lambda')} \sum_{k=1}^{n} f_k(\lambda, \bar{\lambda}') \, g_k(\lambda, \bar{\lambda}) \, e^{-F(\lambda)}, \tag{5.14}$$

where f_k and g_k are 2×2 matrix-valued functions and

$$F(\lambda) = -\frac{i}{2}\left(\frac{\alpha}{\lambda}x + \frac{\sigma_3}{\lambda}y + \alpha^2\frac{\sigma_3}{\lambda^2}t\right).$$

The corresponding solutions, as usual [22, 21], can be constructed explicitly. Indeed, substituting (5.14) into (5.1), one gets

$$\frac{\partial\chi(\lambda, \bar{\lambda})}{\partial\bar{\lambda}} = \sum_{k=1}^{n} h_k(x, y, t) \, g_k(\lambda, \bar{\lambda}) \, e^{-F(\lambda)}, \tag{5.15}$$

where

$$h_k(x, y, t) = \iint_C d\lambda \wedge d\bar{\lambda} \, \chi(\lambda, \bar{\lambda}) \, e^{F(\lambda)} f_k(\lambda, \bar{\lambda}). \tag{5.16}$$

Then, using equation (5.11), one finds

$$\chi(\lambda, \bar{\lambda}, x, y, t) = 1 + \sum_{k=1}^{n} h_k(x, y, t) \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i} \frac{g_k(\lambda', \bar{\lambda}') \, e^{-F(\lambda')}}{\lambda' - \lambda}. \tag{5.17}$$

The system of equations for determining the quantities h_k can be obtained by the multiplication of (5.17) by $e^{F(\lambda)} f_l(\lambda, \bar{\lambda})$ and integrating over λ. One has

$$h_l + \sum_{k=1}^{n} h_k A_{kl} = \xi_l, \quad l = 1, \ldots, n, \tag{5.18}$$

where

$$\xi_l(x, y, t) = \iint_C d\lambda \wedge d\bar{\lambda} \, e^{F(\lambda)} f_l(\lambda, \bar{\lambda}) \tag{5.19}$$

and

$$A_{kl}(x, y, t) = \iint_C \frac{d\lambda \wedge d\bar{\lambda}}{2\pi i} \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda - \lambda'} \times$$

$$\times g_k(\lambda', \bar{\lambda}') \, e^{F(\lambda) - F(\lambda')} f_l(\lambda, \bar{\lambda}); \quad k, l = 1, \ldots, n. \tag{5.20}$$

So, given functions f_k, g_k, one calculates the quantities h_l via the system (5.18) and then the function χ by the formula (5.17). Further, the formulae (5.9), (5.10), (5.12), (5.13) give the solution of equation (1.1).

Thus, we have the class of exact explicit solutions of equation (1.1) which depend on the $8n$ arbitrary functions of two variables. The simplest solution of this type is of the form $(n=1)$:

$$g = \chi_0^{-1}, \quad C = i\alpha(\chi_0^{-1}\chi_1)_{diag}, \tag{5.21}$$

where

$$\chi_0 = 1 + h(x, y, t) \iint_C \frac{d\lambda \wedge d\bar{\lambda}}{2\pi i} g(\lambda, \bar{\lambda}) \, e^{-F(\lambda)},$$

$$\chi_1 = h(x, y, t) \iint_C \frac{d\lambda \wedge d\bar{\lambda}}{2\pi i \lambda^2} g(\lambda, \bar{\lambda}) \, e^{-F(\lambda)}, \tag{5.22}$$

and

$$h = \xi(1 + A)^{-1},$$

$$\xi(x, y, t) = \iint_C d\lambda \wedge d\bar{\lambda} \, e^{F(\lambda)} f(\lambda, \bar{\lambda}), \tag{5.23}$$

$$A(x, y, t) = \iint_C \frac{d\lambda \wedge d\bar{\lambda}}{2\pi i} \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda - \lambda'} g(\bar{\lambda}, \bar{\lambda}') \, e^{f(\lambda) - F(\lambda')} F(\lambda, \bar{\lambda}), \tag{5.24}$$

where $f(\lambda, \bar{\lambda})$ and $g(\lambda, \bar{\lambda})$ are arbitrary 2×2 matrix-valued functions.

The class of solutions with the functional parameters described above contains the subclass of solutions which corresponds to the choice

$$(g_k)_{\alpha\beta}=\delta(\lambda-\lambda_{\alpha\beta}^{(k)})\,\delta(\bar{\lambda}-\bar{\lambda}_{\alpha\beta}^{(k)})\,. \tag{5.25}$$

For such g_k, one has

$$\chi_{\alpha\beta}(\lambda,\bar{\lambda})=\delta_{\alpha\beta}+\sum_{k=1}^{n}\sum_{\gamma=1}^{2}\frac{(h_k(x,y,t))_{\alpha\gamma}\,e^{-F_{\gamma\beta}(\lambda_{\gamma\beta}^{(k)})}}{\lambda-\lambda_{\gamma\beta}^{(k)}}. \tag{5.26}$$

An interesting feature of (5.26) is that each matrix element of such χ has its own set of poles.

In a similar manner one can construct also the rational-exponential solutions. The solutions of this type correspond to the functions f_k, g_k of the form

$$(f_k)_{\alpha\gamma}=\pi\sum_{\beta=1}^{2}\delta(\lambda-\lambda_{\alpha\beta}^{(k)})\,\delta(\bar{\lambda}-\bar{\lambda}_{\alpha\beta}^{(k)})\,(S_k)_{\beta\gamma}\,,$$

$$(g_k)_{\alpha\beta}=\delta(\lambda-\lambda_{\alpha\beta}^{(k)})\,\delta(\bar{\lambda}-\bar{\lambda}_{\alpha\beta}^{(k)})\,;\qquad \alpha,\beta,\gamma=1,2\,, \tag{5.27}$$

where S_k are constant matrices. We will assume that all $\lambda_{\alpha\beta}^{(k)}$ are distinct. Substituting (5.27) into (5.1), one obtains

$$\frac{\partial\chi_{\alpha\beta}}{\partial\bar{\lambda}}=\sum_{k=1}^{n}(h_k(x,y,t))_{\alpha\gamma}\,\delta(\lambda-\lambda_{\gamma\beta}^{(k)})\,\delta(\bar{\lambda}-\bar{\lambda}_{\gamma\beta}^{(k)})\,e^{-F_{\beta\beta}(\lambda)}, \tag{5.28}$$

where

$$(h_k(x,y,t))_{\alpha\delta}=\pi\chi_{\alpha\beta}(\lambda_{\beta\gamma}^{(k)})\,e^{F_{\beta\beta}(x,y,t,\lambda_{\beta\gamma}^{(k)})}(S_k)_{\gamma\delta}\,. \tag{5.29}$$

Further, the substitution of (5.28), (5.29) into (5.11) gives

$$\chi_{\alpha\beta}(\lambda)=\delta_{\alpha\beta}+\sum_{k=1}^{n}\chi_{\alpha\gamma}(\lambda_{\gamma\delta}^{(k)})\,\frac{\exp\{F_{\gamma\gamma}(\lambda_{\gamma\delta}^{(k)})\}\,(S_k)_{\delta\rho}\exp\{-F_{\beta\beta}(\lambda_{\rho\beta}^{(k)})\}}{\lambda-\lambda_{\rho\beta}^{(k)}}. \tag{5.30}$$

So, the function χ is defined by its special values $\chi_{\alpha\gamma}(\lambda_{\gamma\delta}^{(k)})$. The system of equation for determining $\chi_{\alpha\gamma}(\lambda_{\gamma\delta}^{(k)})$ can be derived from (5.30) by proceeding to the limits $\lambda\to\lambda_{\beta\sigma}^{(i)}$. This system is of the form

$$\chi_{\alpha\beta}(\lambda_{\beta\sigma}^{(i)})=\delta_{\alpha\beta}+\sum_{k=1}^{n}\chi_{\alpha\gamma}(\lambda_{\gamma\delta}^{(k)})\,\frac{e^{F_{\gamma\gamma}(\lambda_{\gamma\delta}^{(k)})}(S_k)_{\delta\rho}\,e^{-F_{\beta\beta}(\lambda_{\rho\beta}^{(k)})}}{\lambda_{\beta\sigma}^{(i)}-\lambda_{\rho\beta}^{(k)}},$$

$$\alpha,\beta,\sigma=1,2;\qquad \sigma\neq\beta, \tag{5.31a}$$

$$\chi_{\alpha\beta}(\lambda_{\beta\beta}^{(i)})=\delta_{\alpha\beta}-\chi_{\alpha\gamma}(\lambda_{\gamma\delta}^{(i)})\,e^{F_{\gamma\gamma}(\lambda_{\gamma\delta}^{(i)})}(S_i)_{\delta\beta}\,e^{-F_{\beta\beta}(\lambda_{\beta\beta}^{(i)})}\,F'_{\beta\beta}(\lambda_{\beta\beta}^{(i)})+$$

$$+\sum_{k\neq i}^{n}\sum_{\rho\neq\beta}\frac{\chi_{\alpha\gamma}(\lambda_{\gamma\delta}^{(k)})\,e^{F_{\gamma\gamma}(\lambda_{\gamma\delta}^{(k)})}(S_k)_{\delta\rho}\,e^{-F_{\beta\beta}(\lambda_{\rho\beta}^{(k)})}}{\lambda_{\beta\beta}^{(i)}-\lambda_{\rho\beta}^{(k)}};\qquad \alpha,\beta=1,2\,, \tag{5.31b}$$

where

$$F'_{\beta\beta}(\lambda_{\beta\beta}^{(i)})=\left.\frac{\partial F_{\beta\beta}(\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_{\beta\beta}^{(i)}}.$$

The system (5.31) is a complete system for the calculation of all $\chi_{\alpha\beta}(\lambda_{\beta\sigma}^{(k)})$. The variables x, y and t arise in this system due to F and F' which are linear on x, y, t. As a result, the quantities $\chi_{\alpha\beta}(\lambda_{\beta\sigma}^{(i)})$ and, hence, the solutions g and C of equation (1.1) given by (5.9), (5.10) are, in general, the rational-exponential functions on x, y, t.

In general, for such solutions the each element of χ has its own set of poles. The solutions constructed in the section 3 are the particular case of this solutions with

$$\lambda_{11}^{(k)}=\lambda_{21}^{(k)},\qquad \lambda_{12}^{(k)}=\lambda_{22}^{(k)},\qquad k=1,\ldots,n\,.$$

Note, that the formulae (2.29), (2.30) and (2.35), (2.36) give us the corresponding solutions with the functional parameters and the rational-exponential solutions of the DS and Ishimori equations.

The formulae (5.27) — (5.31) are obviously generalizable to the $N\times N$ matrix case. On the other hand, in the scalar case the system (5.31) is reduced to the single equation

$$\chi(\lambda_i)=1+\sum_{k\neq i}^{n}\frac{\chi(\lambda_k)\,S_k}{\lambda_i-\lambda_k}-\chi(\lambda_i)\,S_i\,F'(\lambda_i)\,, \tag{5.32}$$

where it is assumed that $[F, S_k]=0$. This system gives rise to the pure rational lumps of the KP and NVN equations. For the first time the system (5.32) has been derived by the use of the nonlocal Riemann — Hilbert problem in [22].

6. LUMP AND EXPONENTIAL SOLUTIONS

Other interesting classes of exact solutions are the pure rational and pure exponential solutions. The nonsingular rational solutions (lumps) are of the special interest (see e. g. [1, 2, 10]). Here we present the solutions of equation (1.1) of these types.

We firstly will consider equation (1.1) in the case $\alpha=i$. It is not difficult to verify that this equation possesses the following static rational solution

$$g(z,\bar{z},t)=\frac{1}{|a|^2+|b\bar{z}+c|^2}\begin{pmatrix} b\bar{z}+c, & -a \\ \bar{a} & , \bar{b}z+c \end{pmatrix}, \tag{6.1}$$

$$\bar{\partial}C_{11}=\bar{\partial}\bar{C}_{22}=\frac{i\,|ab|^2}{|a|^2+|b\bar{z}+c|^2}, \tag{6.2}$$

where $z=\frac{1}{2}(y+ix)$, $\bar{z}=\frac{1}{2}(y-ix)$ and a, b, c are arbitrary complex constants. This solution is obviously nonsingular and decays as $|z|^{-1}$ at $|z|\to\infty$, i. e. it is lump. Note that for the solution (6.1) one has $gg^{+}=(|a|^2+|b\bar{z}+c|^2)^{-1}=\det g$.

The solution (6.1) generates the corresponding lump solutions for the DS-II and Ishimori-I equations. For the DS-II equation (equation (2.24) with $\alpha=i$) this static lump is

$$P=\begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}=\frac{1}{|a|^2+|b\bar{z}+c|^2}\begin{pmatrix} 0 & , & iab \\ -i\bar{a}\bar{b}, & 0 \end{pmatrix}, \tag{6.3}$$

$$\varphi \overset{def}{=} \operatorname{tr}(\sigma_3 Q_D)=-\frac{\bar{b}^2(b\bar{z}+c)^2+b^2(\bar{b}z+\bar{c})^2}{(|a|^2+|b\bar{z}+c|^2)^2}. \tag{6.4}$$

Both the fields q and φ decay as $|z|^{-2}$ at $|z|\to\infty$. Emphasize that this static solution of the DS-II equation is essentially differed from the moving lump constructed in [31] for which $|q|\to$const at $|z|\to\infty$.

For the Ishimori-I equation ($\alpha=i$) the lump (6.1) gives rise to the solution

$$S_{+}=S_1+iS_2=\frac{2\bar{a}(b\bar{z}+c)}{|a|^2+|b\bar{z}+c|^2}, \tag{6.5}$$

$$S_3=\frac{|a|^2-|b\bar{z}+c|^2}{|a|^2+|b\bar{z}+c|^2},$$

$$\Phi=2\ln(|a|^2+|b\bar{z}+c|^2). \tag{6.6}$$

The solution (6.5) is nothing but the real vortex-type solution of the Ishimori-I equation found in [30].

The solution (6.1), (6.2), seems, is the simplest nonsingular rational solution of equation (1.1). It would be of interest to construct the general multi-lump solutions of equation (1.1) which

would generate the multi-lump solutions of the DS-II equation and the multi-vortex solutions of the Ishimori-I equation found in [30].

The rational solution similar to (6.1) can be constructed also for equation (1.1) in the case $\alpha=1$. But it is a singular one.

An interesting class of the exact solutions of equation (1.1) at $\alpha=1$ of the different type can be constructed by the use of the coincidence of equation (1.1) with the equation for the DS eigenfunction. Namely, the DS eigenfunctions Ψ_{DS} found in [25] give the exact solutions of equation (1.1). The simplest from these solutions is of the form (see the DS eigenfunction (15) with $k=0$ in [25])

$$g(x,y,t)=1-\frac{1}{\Delta}\begin{vmatrix} \rho\eta\, e^{-\lambda(x+y)}\operatorname{ch}\mu(x-y), & \frac{\lambda}{\mu}\eta\, e^{-2i(\mu+\lambda^2)t} \\ -\frac{\mu}{\lambda}\rho\, e^{2i(\mu^2+\lambda^2)t}, & \rho\eta\, e^{-\mu(x-y)}\operatorname{ch}\lambda(x+y) \end{vmatrix} \tag{6.7}$$

where ρ, η are arbitrary complex constants, λ, μ, are arbitrary real constants and

$$\Delta=e^{\lambda(x+y)+\mu(x-y)}+\rho\eta\operatorname{ch}\mu(x-y)\operatorname{ch}\lambda(x+y). \tag{6.8}$$

The solution (6.7) tends to the constant diagonal matrices along the directions $x+y=0$, $x-y=0$ and $g\to 1$ as $x^2+y^2\to\infty$ outside these lines.

For the DS-I-equation the solution (6.7) generates the exponentially localized breather solution $P_{DS}=\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$ found for the first time in [25].

For the Ishimori-II-equation the solution (6.7) gives rise to the following solution

$$S_{+}=S_1+iS_2=(\det g)^{-1}\frac{\mu}{\lambda}\rho\frac{e^{2i(\mu^2+\lambda^2)t}}{\Delta}\left[1-\rho\eta\frac{e^{-\lambda(x+y)}\operatorname{ch}\mu(x-y)}{\Delta}\right], \tag{6.9}$$

$$S_3=(\det g)^{-1}\left[\left(1-\frac{\rho\eta\, e^{-\lambda(x+y)}\operatorname{ch}\mu(x-y)}{\Delta}\right)\left(1-\frac{\rho\eta}{\Delta}e^{-\mu(x-y)}\operatorname{ch}\lambda(x+y)\right)-\frac{\eta\rho}{\Delta^2}\right],$$

where

$$\det g=1-\frac{\rho\eta}{\Delta}\operatorname{ch}[\mu(x-y)-\lambda(x+y)].$$

For real ρ, η the solution (6.9) is a real one. The solution (6.9) is an exponentially localized breather for the Ishimori-II equation.

Using the more complicated DS eigenfunctions one is able to construct the more complicated solutions of equations (1.1) and the Ishimori equation.

7. OTHER (2+1)- AND (1+1)-DIMENSIONAL INTEGRABLE EQUATIONS FOR EIGENFUNCTIONS

1. Nonlinear integrable equations for the eigenfunctions can be written done in the other cases too. It is a rather general phenomenon.

Here we will consider the corresponding equations associated with the N-waves and KP-equations.

The (2+1)-dimensional resonantly interacting waves equation is of the form (see [1, 2])

$$P_t(x,y,t)+V^xP_x+V^yP_y-[V^yP,P]=0, \tag{7.1}$$

where P is the $N\times N$ matrix $(P_{ii}=0)$, $N\geqslant 3$,

$$V^x\cdot=-A\,\mathrm{ad}_A^{-1}[B,\cdot]+B,\qquad V^y\cdot=-\mathrm{ad}_A^{-1}[B,\cdot],$$

A, B are diagonal matrices and $\mathrm{ad}_A\Phi\overset{def}{=}[A,\Phi]$. Equation (7.1) is the compatibility condition for the linear system [1, 2]

$$\frac{\partial\Psi}{\partial y}+A\frac{\partial\Psi}{\partial x}+P\Psi=0, \tag{7.2a}$$

$$\frac{\partial\Psi}{\partial t}+B\frac{\partial\Psi}{\partial x}+(\mathrm{ad}_A^{-1}[B,P])\,\Psi=0. \tag{7.2b}$$

Excluding the potential P from (7.2), we arrive at the nonlinear equation for the eigenfunction Ψ:

$$\Psi_t+B\Psi_x-(\mathrm{ad}_A^{-1}[B,(\Psi_y+A\Psi_x)\,\Psi^{-1}])\,\Psi=0. \tag{7.3}$$

This equation becomes linear in the terms of the left currents:

$$\Psi_t\Psi^{-1}+B\Psi_x\Psi^{-1}-\mathrm{ad}_A^{-1}[B,\Psi_y\Psi^{-1}+A\Psi_x\Psi^{-1}]=0, \tag{7.4}$$

or

$$[A,\Psi_t\Psi^{-1}+B\Psi_x\Psi^{-1}]-[B,\Psi_y\Psi^{-1}+A\Psi_x\Psi^{-1}]=0. \tag{7.5}$$

Equation (7.3) is equivalent to the compatibility condition for the system

$$L_1\tilde{\Psi}\overset{def}{=}(\Psi\partial_y+A\Psi\partial_x)\,\tilde{\Psi}=0, \tag{7.6a}$$

$$L_2\tilde{\Psi}\overset{def}{=}(\Psi\partial_t+B\Psi\partial_x+\Psi)\,\tilde{\Psi}=0 \tag{7.6b}$$

with the operator representation

$$[L_1,L_2]=\gamma_1L_1+\gamma_2L_2, \tag{7.7}$$

where

$$\gamma_1=[\Psi,B]\,\partial_x-\Psi\,\mathrm{ad}_A^{-1}[B,\Psi_y\Psi^{-1}+A\Psi_x\Psi^{-1}],$$

$$\gamma_2=[A,\Psi]\,\partial_x+\Psi\,(\Psi_y+A\Psi_x)\,\Psi^{-1}.$$

Equation (7.3) is the generating equation. Evidently, introducing the variable $P=-(\Psi_y+A\Psi_x)\Psi^{-1}$, we arrive at equation (7.1). Second equation is generated by the variable $S=\Psi^{-1}A\Psi$. This equation for S can be written down in a closed form if all the diagonal elements of A are distinct. In this case the arbitrary diagonal matrix B can be represented in the form

$$B=\sum_{n=0}^{N-1}c_nA^n,$$

where c_n are some constants (see e. g. [32]) and, hence,

$$\Psi^{-1}B\Psi=\sum_{n=0}^{N-1}c_nS^n.$$

The corresponding equation for S is easily derived from (7.4) and looks like

$$S_t(x,y,t)+\sum_{n=0}^{N-1}c_n(S^nS_x-(S^n)_y-S(S^n)_x)=0. \tag{7.8}$$

Equation (7.8) is equivalent to the equation $[L_1^S,L_2^S]=0$ *with*

$$L_1^S=\partial_y+S\partial_x,$$

$$L_2^S=\partial_t+\sum_{n=0}^{N-1}c_nS^n\partial_x. \tag{7.9}$$

Thus, each solution of equation (7.3) generates the solutions of the equations (7.1) and (7.8).

Now let us consider another example, namely, the KP-equation

$$(u_t+u_{xxx}+6uu_x)_x+3\alpha^2u_{yy}=0, \tag{7.10}$$

where $u(x, y, t)$ is the scalar function and $\alpha^2 = \pm 1$. The KP-equation (7.10) is the compatibility condition for the linear system (see e. g. [1, 2])

$$L_1 \Psi = (\alpha \partial_y + \partial_x^2 + u) \Psi = 0, \tag{7.11a}$$

$$L_2 \Psi = (\partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x - 3\alpha W) \Psi = 0, \tag{7.11b}$$

where $W_x = u$. From (7.11a) one has

$$u = -(\alpha \Psi_y + \Psi_{xx}) \Psi^{-1}. \tag{7.12}$$

Substituting this expression for u into (7.11b), we arrive at the following equation for the eigenfunction Ψ:

$$\Psi_t + \Psi_{xxx} - 3\Psi_x \Psi_{xx} \Psi^{-1} - 3\alpha \Psi_x \Psi_y \Psi^{-1} - 3\alpha \Psi_{xy} - 3\alpha \Psi W_y = 0, \tag{7.13a}$$

$$W_x + \alpha \Psi_y \Psi^{-1} + \Psi_{xx} \Psi^{-1} = 0. \tag{7.13b}$$

Equation (7.13) is equivalent to the compatibility condition for the linear system

$$L_1^{\Psi} \tilde{\Psi} = (\Psi \partial_y + \Psi \partial_x^2 + 2\Psi_x \partial_x) \tilde{\Psi} = 0,$$

$$L_2^{\Psi} \tilde{\Psi} = (\Psi \partial_t + 4\Psi \partial_x^3 + 12\Psi_x \partial_x^2 - 6(\alpha \Psi_y - \Psi_{xx}) \partial_x) \tilde{\Psi} = 0 \tag{7.14}$$

with the quartet operator representation $[L_1^{\Psi}, L_2^{\Psi}] = \gamma_1 L_1^{\Psi} + \gamma_2 L_2^{\Psi}$ with

$$\gamma_1 = -12\Psi_x \partial_x^2 - 12\Psi_{xx} \partial_x - 3\Psi_{xxx} +$$
$$+ 3\Psi_{xx} \Psi_x \Psi^{-1} + 3\alpha \Psi_y \Psi_x \Psi^{-1} - 3\alpha \Psi_{xy} - 3\alpha W_y \Psi, \tag{7.15}$$
$$\gamma_2 = 2\Psi_x \partial_x + \alpha \Psi_y + \Psi_{xx}.$$

In the terms of the left currents equation (7.13) looks like

$$\Psi_t \Psi^{-1} + (\Psi_x \Psi^{-1})_{xx} + 2(\Psi_x \Psi^{-1})^3 -$$
$$- 3\alpha (\Psi_x \Psi^{-1})_y - 6\alpha (\Psi_x \Psi^{-1})(\Psi_y \Psi^{-1}) - 3\alpha W_y = 0, \tag{7.16a}$$

$$W_x = -\alpha \Psi_y \Psi^{-1} - (\Psi_x \Psi^{-1})_x - (\Psi_x \Psi^{-1})^2. \tag{7.16b}$$

In this scalar case equation (7.16) can be essentially simplified. Indeed, introducing the variable $v = (\ln \Psi)_x$, one gets from (7.16a) the equation

$$v_t + v_{xxx} + 6v^2 v_x - 6\alpha v_x \partial_x^{-1} v_y + 3\alpha^2 \partial_x^{-1} v_{yy} = 0. \tag{7.17}$$

Equation (7.17) is nothing but the modified KP-equation found in [33, 34]. On the other hand the relation (7.16b) becomes

$$u = -v_x - v^2 - \alpha \partial_x^{-1} v_y \tag{7.18}$$

that is, just the two-dimensional Miura transformation [33, 34].

So, in the KP-case the nonlinear equation for the eigenfunction Ψ is, in fact, equivalent to the modified KP-equation ($\Psi = e^{\partial_x^{-1} v}$).

This equivalence gives us the possibility to construct the solutions of the modified KP-equation using the known KP-eigenfunctions Ψ. In particular, the solution of equation (7.3) at $\alpha = i$ of the form

$$\Psi = e^{ikx - ik^2 y} \left(1 + \frac{i}{k - \lambda} \frac{\xi + \mu}{|\xi|^2 + \mu^2} + \frac{i}{k - \bar{\lambda}} \frac{\xi - \mu}{|\xi|^2 + \mu^2} \right),$$

where $\mu = i(\lambda - \bar{\lambda})^{-1}$ and $\xi \overset{def}{=} x - 2\lambda y + 12\lambda^2 t + \gamma$ and k, λ, γ are arbitrary complex constants (i. e. eigenfunction Ψ which corresponds to the one-lump solution of the KP-I-equation, (see [1, 2])) gives rise to the pure rational solution $v = \Psi_x / \Psi$ of the modified KP-equation.

The solutions of equation (7.13), of course, generates the solutions of the KP-equation by the formula (7.12).

The second equation, similar to the Ishimori equation or equation (7.8) is, obviously, absent.

2. In the one-dimensional limit $g_y \equiv 0$ equation (1.1) is reduced to the simple equation

$$i g_t - \sigma_3 g_{xx} + 2\sigma_3 g_x g^{-1} g_x = 0 \tag{7.19}$$

or

$$i g_t g^{-1} - \sigma_3 (g_x g^{-1})_x + \sigma_3 (g_x g^{-1})^2 = 0. \tag{7.20}$$

Equation (7.19) is equivalent to the operator quartet equation

$$[L_1^{|+|}, L_2^{|+|}] = \gamma_1^{|+|} L_1^{|+|} + \gamma_2^{|+|} L_2^{|+|}$$

with

$$L_1^{|+|} = -\sigma_3 g \partial_x, \quad L_2^{|+|} = i g \partial_t, \tag{7.21}$$
$$\gamma_1^{|+|} = g\sigma_3 (g_x g^{-1})_x + g\sigma_3 (g_x g^{-1})^2,$$
$$\gamma_2^{|+|} = [g, \sigma_3] \partial_x - g\sigma_3 g_x g^{-1}.$$

Note that in the term of the variable $G = g^{-1}$ equation (7.19) looks like

$$i G_t - G\sigma_3 G^{-1} G_{xx} = 0.$$

The formulae for the inverse spectral transform for equation (7.19) can be obtained as the reduction of the corresponding formulae for equation (1.1). It is easy to check that if g obeys equation (7.19) then the variable $S=-g^{-1}\sigma_3 g$ obey the Heisenberg ferromagnet model equation

$$iS_t+\frac{1}{2}[S,S_{xx}]=0 \tag{7.22}$$

or $\vec{S}_t+\vec{S}\times\vec{S}_{xx}=0$ (see e.g. [1, 2, 6]). At the same time the variable $P=\sigma_3 g_x g^{-1}$ obeys the NLS-equation (see e. g. [1, 2])

$$iP_t-\sigma_3 P_{xx}-\sigma_3 \operatorname{tr} P^2 P=0, \tag{7.23}$$

or

$$P=\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix},$$

$$iq_t-q_{xx}+2qrq=0,$$

$$ir_t+r_{xx}-2qrr=0.$$

The auxiliary linear system for the NLS-equation is (see e. g. [1, 2])

$$L_1^{\text{NLS}}\Psi_{\text{NLS}}=(-\sigma_3\partial_x+P+\mu)\Psi_{\text{NLS}}=0,$$

$$L_2^{\text{NLS}}\Psi_{\text{NLS}}=(i\partial_t-2\sigma_3\partial_x^2+2P\partial_x+P_x-\sigma_3P^2)\Psi_{\text{NLS}}=0. \tag{7.24}$$

where μ is a spectral parameter.

A nonlinear equation for the entire eigenfunction Ψ_{NLS} which follow from (7.24) is of the form

$$i\Psi_{\text{NLS}_t}-\sigma_3\Psi_{\text{NLS}_{xx}}+2\sigma_3\Psi_{\text{NLS}_x}\Psi_{\text{NLS}}^{-1}\Psi_{\text{NLS}_x}-2\mu\Psi_{\text{NLS}_x}-\sigma_3\mu^2\Psi_{\text{NLS}}=0. \tag{7.25}$$

It is easy to see that equation (7.19) coincides with the equation (7.25) for Ψ_{NLS} $(\mu=0)$. So, the situation with the generating equation (7.19) and the integrable equation for the NLS-eigenfunction Ψ_{NLS} is similar to the (2+1)-dimensional case.

Note also that the reduction $gg^{+}=1$ is admissible by equation (7.19) which under this reduction looks like

$$ig_t-\sigma_3 g_{xx}+2\sigma_3 g_x g^{+} g_x=0. \tag{7.26}$$

The corresponding spin variable $\vec{S}$ for equation (7.22) is real in this case and one has $P=\begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}$ for the NLS-equation.

In conclusion let us consider the (1+1)-dimensional analogs of equations (7.3) and (7.13). They are

$$\Psi_t+B\Psi_x-[B,A\,\operatorname{ad}_A^{-1}(\Psi_x\Psi^{-1})]\Psi=0 \tag{7.27}$$

for (6.3) and

$$\Psi_t+\Psi_{xxx}-3\Psi_x\Psi_{xx}\Psi^{-1}=0. \tag{7.28}$$

for the Korteweg-de Vries (KdV) case. Equation (7.28) coincides with the nonlinear equation for the zero spectral parameter value $\Psi_{\text{KdV}}(\lambda=0)$ of the KdV-eigenfunction $\Psi_{\text{KdV}}(\lambda)$ which obeys the linear system [1—6]

$$(\partial_x^2+u-\lambda)\Psi_{\text{KdV}}=0,$$

$$(\partial_t+4\partial_x^3+6u\partial_x+3u_x)\Psi_{\text{KdV}}=0. \tag{7.29}$$

The integrable nonlinear equation for the entire eigenfunction $\Psi_{\text{KdV}}(\lambda)$ is of the form

$$\Psi_t+6\lambda\Psi_x+\Psi_{xxx}-3\Psi_x\Psi_{xx}\Psi^{-1}=0. \tag{7.30}$$

Equation (7.30) obviously is reduced to equation (7.28) by the redefinition $\partial_t+6\lambda\partial_x\to\partial_t$.

Equation (7.28) by the formula $u=-\Psi_{xx}\Psi^{-1}$ generates the solutions of the KdV-equation. On the other hand the change $\Psi\to v=\Psi_x/\Psi$ converts it into the modified KdV-equation.

These interrelations and the corresponding Miura transformation $u=-v_x-v^2$, seems, have been the part of the arguments which have led to the discovery of the IST method in [35, 36].

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Б.Г. Конопельченко, Б.Т. Маткаримов

**Обратное спектральное преобразование
для нелинейного уравнения,
порождающего уравнения Дэви—Стюардсона и Ишимори**

Ответственный за выпуск С.Г.Попов

Работа поступила 10 мая 1989 г.
Подписано в печать 4.09 1989 г. МН 02337.
Формат бумаги 60×90 1/16 Объем 2,9 печ.л., 2,4 уч.-изд.л.
Тираж 290 экз. Бесплатно. Заказ № 123.

*Набрано в автоматизированной системе на базе фото-наборного автомата ФА1000 и ЭВМ «Электроника» и отпечатано на ротапринте Института ядерной физики СО АН СССР,
Новосибирск, 630090, пр. академика Лаврентьева, 11.*