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INTERACTION OF CLASSICAL YANG-MILLS CHARGES AND THE PROBLEM OF QUARK CONFINEMENT

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abstract

Equations and boundary conditions for the field created by two point-like Yang-Mills charges at rest are obtained. Non-trivial property of this static system is the existence of magnetic field. The connection of the model considered with the problem of quark confinement is discussed.

It is well known that the classical solution for the field of point-like Yang-Mills charge is reduced to the usual Coulomb form in spite of formal non-linearity of equations. It is useful to trace the origin of this fact. Write down the equations of Yang-Mills field for the case when its source has a time component only:

$$(\delta^{\alpha\gamma}\partial_n - g\varepsilon^{\alpha\beta\gamma} b_n^\beta)(\partial_n b_0^\gamma + \partial_0 b_n^\gamma + g\varepsilon^{\gamma\delta\varepsilon} b_0^\delta b_n^\varepsilon) = \rho^\alpha \qquad (1)$$

$$(\delta^{\alpha\gamma}\partial_0 + g\varepsilon^{\alpha\beta\gamma} b_0^\beta)(\partial_n b_0^\gamma + \partial_0 b_n^\gamma + g\varepsilon^{\gamma\delta\varepsilon} b_0^\delta b_n^\varepsilon) + \qquad (2)$$
$$+ (\delta^{\alpha\gamma}\partial_m - g\varepsilon^{\alpha\beta\gamma} b_m^\beta)(\partial_n b_m^\gamma - \partial_m b_n^\gamma + g\varepsilon^{\gamma\delta\varepsilon} b_m^\delta b_n^\varepsilon) = 0$$

All the terms in the left-hand side of the equ.(1) except Δb_0^α contain space components of the vector-potential b_n^α. In the equ.(2) the only term in the static case not turning to zero at $b_n^\alpha = 0$ is $g\varepsilon^{\alpha\beta\gamma} b_0^\beta \partial_n b_0^\gamma$. It is clear therefore that if the direction of the field b_0^α in the isotopic space does not depend on coordinates (and just this situation takes place evidently in the case of one charge when $\rho^\alpha = g t^\alpha \delta(\vec{r})$, then that term turns to zero so that $b_n^\alpha = 0$ is the solution of the equ. (2). Simultaneously the equ.(1) is reduced to the usual Poisson form and the problem as a whole to the trivial Coulomb one.

At the first sight, it follows immediately from this consideration that to get non-trivial static solutions one should mix explicitly isotopic and space indices, i.e., to look for the solutions, say, in the form $b_0^\alpha = t^\alpha f(r)$. However, more simple way out exists: it is sufficient to consider the field created by two, and not one, centres. It can be easily seen that in this case the direction of the field b_0^α in the isotopic space depends on coordinates, generally speaking, in non-trivial way, so that the problem becomes effectively non-linear.

Therefore, consider the field created by two point-like Yang-Mills charges at rest. To get maximal simplification of the problem assume further that the isotopic spins of the sources as well as their geometric sum are so large that both isospins may be considered as classical vectors. The source in the right-hand side of the equ.(1) is presented as

$$\rho^\alpha = g t\left[\bar{i}_1^\alpha \delta(\bar r - \bar r_1) + \bar{i}_2^\alpha \delta(\bar r - \bar r_2)\right] \qquad (3)$$

where t is the modulus of the vector of the source isospin (we take for convenience that it is the same for both particles), $\bar{i}_{1,2}^\alpha$ are the orts of these vectors.

Look for the solution of the system (1)-(3) in the form

$$b_0^\alpha = \frac{1}{g}\left[\bar{i}_1^\alpha(t)\, b^1(\bar r) + \bar{i}_2^\alpha(t)\, b^2(\bar r)\right] \qquad (4)$$

$$b_n^\alpha = \frac{1}{g}\varepsilon^{\alpha\beta\gamma}\,\bar{i}_1^\beta(t)\,\bar{i}_2^\gamma(t)\, b_n(\bar r) \qquad (5)$$

The equations of motion for the vectors $\bar{i}_{1,2}^\alpha(t)$ are evident:

$$\dot{\bar{i}}_{1,2}^\alpha = -g\varepsilon^{\alpha\beta\gamma} b_0^\beta(\bar r_{1,2})\,\bar{i}_{1,2}^\gamma \qquad (6)$$

in other words, the isotopic spin is precessing around the vector b_0^α. The equ.(6) can be easily seen to guarantee the vanishing of the generalized divergence of external current:

$$\partial_\mu j_\mu^\alpha + g\varepsilon^{\alpha\beta\gamma} b_\mu^\beta j_\mu^\gamma = 0 \qquad (7)$$

The condition (7) is known to follow from the field equations (1),(2), so that the equ.(6) is necessary for the self-consistency of the problem. Substituting (4) into (6) we get

$$\dot{\bar{i}}_1^\alpha = b^2(\bar r_1)\,\varepsilon^{\alpha\beta\gamma}\,\bar{i}_1^\beta\,\bar{i}_2^\gamma, \qquad \dot{\bar{i}}_2^\alpha = -b^1(\bar r_2)\,\varepsilon^{\alpha\beta\gamma}\,\bar{i}_1^\beta\,\bar{i}_2^\gamma \qquad (8)$$

It is evident that in degenerate case when the sources' isospins are parallel or antiparallel, the problem is trivial again.

In the general case the substitution of (4),(5),(8) into

(1) and (2) leads to the following equations:

$$\partial_n \Phi_{1n} + (\theta\Phi_{1n} + \Phi_{2n})\, b_n = 4\pi\beta\,\delta(\bar r - \bar r_1)$$
$$\partial_n \Phi_{2n} - (\Phi_{1n} + \theta\Phi_{2n})\, b_n = 4\pi\beta\,\delta(\bar r - \bar r_2) \qquad (9)$$

$$\partial_m(\partial_m b_n - \partial_n b_m) + \varphi_2\Phi_{1n} - \varphi_1\Phi_{2n} = 0 \qquad (10)$$

where

$$\Phi_{1n} = \partial_n\varphi_1 + (\theta\varphi_1 + \varphi_2)\, b_n, \qquad \Phi_{2n} = \partial_n\varphi_2 - (\varphi_1 + \theta\varphi_2)\, b_n$$

$$\theta = \bar{i}_1^\alpha\,\bar{i}_2^\alpha, \qquad \beta = \frac{g^2}{4\pi}t$$

and

$$\varphi_1(\bar r) = b^1(\bar r) - b^1(\bar r_2), \qquad \varphi_2(\bar r) = b^2(\bar r) - b^2(\bar r_1) \qquad (11)$$

The remarkable fact is that not the time component of the vector-potential by itself (via $b^{1,2}(\bar r)$) enters the field equations (9),(10), but the quantities $\varphi_{1,2}(\bar r)$ defined by the equalities (11). Some hint on the explanation of this fact is that the potential created by one of the sources at the point where another is located, is spent by virtue of the equs.(8) on the rotation of isospin of that, another, source; and therefore it should be subtracted from dynamical quantity characterizing the field in the usual space. The functions $\varphi_{1,2}(\bar r)$ merely due to their definition (11) satisfy the boundary conditions

$$\varphi_1(\bar r_2) = \varphi_2(\bar r_1) = 0 \qquad (12)$$

It should be stressed that these conditions are surely specifical for the Yang-Mills nature of the field.

As to the point-like sources in the right-hand side of the equs.(9), we shall take as usually that they define the singularity of the functions $\varphi_{1,2}$, i.e., they lead to the boundary conditions

$$\varphi_{1,2}(\bar r)\Big|_{\bar r \to \bar r_{1,2}} = -\frac{\beta}{|\bar r - \bar r_{1,2}|} \qquad (13)$$

The behaviour of other functions at $\bar{r} \to \bar{r}_{1,2}$ should be sufficiently smooth so that the terms $\Delta\varphi_{1,2}$ be indeed the most singular ones in the left-hand sides of the equs.(9).

Discuss now the relation between the equations obtained and the gauge invariance of the theory. The equs.(9),(10) follow from the initial ones (1),(2) only if the vectors $\bar{i}_1$, $\bar{i}_2$ and $\bar{i}_1 \times \bar{i}_2$ are linearly independent, i.e., if the isotopic spins of the sources are not parallel. It is clear however that such a condition is not gauge invariant, generally speaking. Indeed, in virtue of this invariance the vectors $\bar{i}_1$ and $\bar{i}_2$, being at different points of the space, can be rotated every by arbitrary angle; in particular, they can be made parallel, and in this case the solution should have, it seems, the trivial Coulomb form.

But in fact the choice of the solution in the form (4),(5) fixes the gauge to a considerable degree. And the angle between $\bar{i}_1$ and $\bar{i}_2$ is fixed due to Gauss theorem by given total isospin $\bar{T}$ of the system and by the coefficients at r^{-1} in the asymptotic expressions for the functions φ_1 and φ_2, of course, if the asymptotics of these functions is reasonable. Thus, the described formulation of the problem is a reasonable one in spite of gauge invariance of the initial equations.

However, the gauge invariance may be used to exclude the procession of isospins described by the equs.(8), i.e., to make $\bar{i}_{1,2}$ constant ones, conserving of course the angle between them. Then the boundary conditions (12) arise simply from the requirement of the self-consistency of the system of equations (9),(10). One can check it easily substituting the equs.(9) into the divergence of the vector equation (10).

It should be noted also that the system of the equations (9),(10) is invariant under the transformations with the infinitesimal form

$$\delta\varphi_1 = \delta\alpha(\theta\varphi_1 + \varphi_2), \qquad \delta\varphi_2 = -\delta\alpha(\varphi_1 + \theta\varphi_2)$$
$$\delta b_n = -\partial_n \delta\alpha \tag{14}$$

The additional requirement on these transformation is that they should not change the boundary conditions (12),(13). Due to this invariance the vector-potential b_n can be chosen in different ways. The Coulomb gauge $\partial_n b_n = 0$ is convenient and will be used below.

Investigate the solution near one of the centers, say, the first one. Since the function φ_1 has here the singularity: $\varphi_1 = -\frac{\beta}{\rho}$, $\rho = |\bar{r} - \bar{r}_1|$, the problem can be linearised in respect to other quantities. For the solution it is convenient to introduce the magnetic field vector $\bar{h} = \bar{\nabla} \times \bar{b}$ which has in spherical coordinates only one component, the asimuthal one. In the mentioned linear approximation one can easily get for it from (10) the following equation:

$$\bar{\nabla} \times \rho^2 [\bar{\nabla} \times \bar{h}] - \beta^2 \bar{h} = 0 \tag{15}$$

Its total solution is

$$h(\rho,\theta) = \sum_{\ell \geq 1} P_\ell^1(\theta) \rho^{-3/2} \left[c_\ell^1 \rho^{\sqrt{(\ell+1/2)^2 - \beta^2}} + c_\ell^2 \rho^{-\sqrt{(\ell+1/2)^2 - \beta^2}} \right] \tag{16}$$

From the same equation (10) in the Coulomb gauge where $\partial_n b_n = 0$, we get

$$\Delta(\rho\varphi_2) = \beta^{-1} \bar{\nabla} [\rho^2 \bar{\nabla} \times \bar{h}] \tag{17}$$

One can easily find from here that the condition (12) is satisfied only if all $c_\ell^2 = 0$ and c_ℓ^1 do not vanish at

$$\lambda = \sqrt{(\ell + 1/2)^2 - \beta^2} > \frac{1}{2} \tag{18}$$

Standard calculations show further that the total solution in the vicinity of the point $\bar{r}_1$ looks as follows:

$$\begin{aligned}
\varphi_1 &= \beta\left(a - \frac{1}{\rho}\right) \\
\varphi_2 &= \beta \cdot 2b\,\ell(\ell+1)\rho^{\lambda - 1/2} P_\ell(\theta) \\
b_r &= -\beta^2 \cdot b\,\ell(\ell+1)\rho^{\lambda - 1/2} P_\ell(\theta) \\
b_\theta &= \beta^2 \cdot b[2\ell(\ell+1) + (\beta^2+2)(\lambda - \tfrac{1}{2})]\rho^{\lambda - 1/2} P_\ell^1(\theta) \\
h &= \beta^2 \cdot b[4\ell(\ell+1) - \beta^2(\beta^2+2)]\rho^{\lambda - 3/2} P_\ell^1(\theta)
\end{aligned} \tag{19}$$

The coupling constant β is extracted from the integration constants a and b in such a way that it should be clear in what order of perturbation theory in β the corresponding function arises.

Two integration constants a and b are sufficient to define in the vicinity of the point $\bar{z}_1$ the values of functions and their partial derivatives of the first order. (Since the function φ_1 is singular here, relative to it one should talk about the values of $\rho\varphi_1$ and partial derivatives of $\rho\varphi_1$.) Therefore the solution in other regions obtained by the continuation from the mentioned vicinity, will be also fully determined by the two parameters a and b. The values of these constants are fixed by means of two more boundary conditions (12),(13) at the point $\bar{z}_2$. Thus, the equs.(9),(10) together with the boundary conditions (12),(13) define the solution of the problem completely. We wish to stress that as it follows from the condition (18), that the minimal value of ℓ in the multipole expansion of the solution near the source, and hence the qualitative behaviour of the solution as a whole, is dependent on the magnitude of the coupling constant.*)

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*) Formally the situation resembles that arising when one solves the problem of the behaviour of small deviation of Yang-Mills field from the Coulomb solution corresponding to a single point-like center /1/. But the singularity of the vector field in the Coulomb potential at sufficieatly large coupling constant, or fall to the center, found in the work /1/, is in no way specific to the Yang-Mills problem. This phenomenon is well-known for the usual relativistic Coulomb problem and does not depend in a renormalizable theory on the spin of the particle (here a renormalizable interaction of the vector particle is discussed). As to the dependence of the character of the solution on the magnitude of coupling constant, found in the present work, it is a specifical Yang-Mills phenomenon , surely it has no electrodynamic analogue.

There is a beautiful physical analogue for the problem considered. By means of the substitution

$$\varphi_1 = \frac{1}{2z_+ z_-}\left[(z_+ + z_-)\psi_1 - (z_+ - z_-)\psi_2\right]$$
$$\varphi_2 = \frac{1}{2z_+ z_-}\left[-(z_+ - z_-)\psi_1 + (z_+ + z_-)\psi_2\right] \qquad (20)$$
$$\bar{b} = \frac{\bar{a}}{z_+ z_-}, \qquad z_\pm = \sqrt{1\pm\theta}$$

the system of equs.(9),(10) may be reduced to the form

$$(-i\bar\nabla - \bar{a})^2\psi = 0, \qquad \psi = \frac{\psi_1 + i\psi_2}{\sqrt{2}} \qquad (21)$$
$$\bar\nabla\times\bar{H} = -\bar{j}, \qquad \bar{H} = \bar\nabla\times\bar{a}, \qquad (22)$$
$$\bar{j} = \psi^*(-i\bar\nabla - \bar{a})\psi + \left[(-i\bar\nabla - \bar{a})\psi\right]^*\psi$$

The omitted δ-function sources in the right-hand side of the equ.(21) are taken into account by means of evident boundary conditions on the function ψ . We have come in this way to the electrodynamics of the scalar field ψ in three-dimensional space. The only distinction is the inverse, in comparison with the usual one, sign of the current in the Maxwell equation (22). Its cause can be easily understood: usually the terms $|(-i\bar\nabla - \bar{a})\psi|^2$ and $\frac{1}{2}\bar{H}^2$ enter the Lagrangian with the same sign, but in our case when the first term is the transformation of $\frac{1}{2}f^{\alpha}_{\mu\nu}f^{\alpha}_{\mu\nu}$, and the second one is the transformation of $\frac{1}{4}f_{\mu\nu}f^{\alpha}_{\mu\nu}$, their signs in the Lagrangian are opposite. This distinction in sign is of importance. The asimuthal magnetic field created by the currents does not tend to pinch these currents as usually, it pushes them apart. So to say, anti-pinch-effect arises. The presence of magnetic field in the static problem of interaction of Yang-Mills charges is by itself, from our point of view, a very interesting fact. It is not surprising in this situation that even qualitative picture of the phenomenon changes as the coupling constant increases.

Unfortunately, complete solution of the problem still is not found. It is not even evident that a reasonable solution exists at all. As it was mentioned above, the equs.(9),(10) and the boundary conditions (12),(13) define uniquely in general the solution in a finite region containing both centers. But it is not clear generally speaking whether this solution turns at $r \to \infty$ into a well-decreasing asymptotic solution that corresponds in virtue of Gauss theorem to the given value of total isospin of the system. If a reasonable solution is in fact absent and in addition the configuration with parallel isospins is unstable, does not it mean in the language of quantum chromodynamics, i.e., when passing from the group SU(2) to the group SU(3), that only "white" states are realized in nature?*)

If the solution exists, the energy of interaction of the charges considered depends on the distance between them as $|\vec{r}_1 - \vec{r}_2|^{-1}$. It follows immediately from dimensional considerations. It is not clear however whether this interaction is attractive or repulsive. Some consideration in the favour of repulsion at sufficiently large coupling constant is the fact that the magnetic field energy, and it is always positive, increases with β in perturbation theory faster than the electrostatic one. However, it can be in no way excluded that the interaction is attractive. If the attraction increases fastly with the growth of the coupling constant, such a situation can serve as indication of quark confinement in a less simplified model of interaction.

The fact that the model considered is both natural and nontrivial justifies, as it seems to me, the publication of this work, although it does not contain definite conclusions.

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*) This possibility to interprete the absence of solution was pointed to me by V.N.Gribov.

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