Siberian Branch of Russian Academy of Science
BUDKER INSTITUTE OF NUCLEAR PHYSICS

B.V. Chirikov

CRITICAL PERTURBATION IN STANDARD MAP: A BETTER APPROXIMATION

Budker INP 98-75

http://www.inp.nsk.su/publications

Novosibirsk
1998

Critical perturbation in standard map:
A better approximation

B.V. Chirikov

Budker Institute of Nuclear Physics
630090 Novosibirsk, Russia

Abstract

Direct computation of the transition time between neighbor resonances in the standard map, as a function of the perturbation parameter K, allows for improving the accuracy of the critical perturbation value up to $K_{cr} - K_g < 2.5 \times 10^{-4}$ that is by a factor of about 50 as compared to the previous result due to MacKay and Percival.

Email: chirikov@inp.nsk.su

As is well known by now a typical structure of the phase space of a few–freedom nonlinear dynamical system is characterized by a very complicated admixture of both chaotic as well as regular (integrable) components of motion (the so–called divided phase space, see, e.g., [1, 2, 3]). Statistical properties of such a motion are very intricate and unusual. One of the most interesting (and important for many applications) problem is the conditions for transition from a local (restricted to relatively small regions in phase space) to the global chaos covering the whole available phase space. The most studied model of such a transition is described by the so–called (canonical) standard map (for history of this model see [4]):

$$\overline{y} = y - \frac{K}{2\pi} \cdot \sin(2\pi x) \ , \quad \overline{x} = x + \overline{y} \tag{1}$$

where K is the perturbation parameter. In this simple model the transition to global chaos corresponds to some exact critical value $K = K_{cr}$. For $K > K_{cr}$ the motion becomes infinite (in momentum y) for some initial conditions while for $K \le K_{cr}$ all the trajectories are confined within a period of map (1): $\Delta y = 1$.

The first idea how to solve this difficult problem was due to Greene [5]. First, he was able to solve a much simpler problem of the critical perturbation $K(r)$ at which a particular invariant Kolmogorov - Arnold - Moser (KAM) curve with the rotation number r is destroyed. Critical function $K(r)$ is extremely singular with big dips at everywhere dense set of rational r values (see, e.g., [6]). The physical mechanism of this behavior (known since Poincaré) is explained by resonances in the system (1) as the rotation number is the ratio of oscillation/perturbation frequencies. Whence, the main Greene's idea: to find the 'most irrational' $r = r_g$ which would correspond to the motion 'most far–off' all the resonances. The former is well known in the number theory: $r_g = [111...] = (\sqrt{5} - 1)/2$ where the first representation is a continued fraction. This 'golden' curve was found to be critical at the parameter $K = K_g = 0.97163540631...$ [5, 7]. It was conjectured that for $K > K_g$ all invariant curves are destroyed [7], that is $K_{cr} = K_g$.

The 'most–irrational' assumption - as plausible as it is - remains a hypothesis. The main difficulty is here in that the resonance interaction and overlap, destroying invariant curves, depend not only on the resonance spacings, which are indeed maximal for $r = r_g$, but also on the amplitudes of those which are not simply an arithmetical property. Another argument, based on the analysis of the critical function $K(r)$ [8, 9], also does not prove this principal hypothesis.

A different approach to the problem - the so–called converse KAM theory - was developed in [10, 11]. It relies upon a rigorous criterion for the absence of any invariant curve in a certain region. Unfortunately, this criterion can only be checked numerically, and besides it provides the upper bound K_{cr}^+ only (the lower bound $K_{cr}^- = K_g$). The remaining gap, or the accuracy of K_{cr}:

$$(\Delta K)_{cr} = K_{cr}^+ - K_{cr} \qquad (2)$$

can be made arbitrarily small at the expense of computation time t_C which scales as [10]

$$t_C \propto (K_{cr}^+ - K_{cr})^{-p} \qquad (3)$$

Facing this difficulty, it is natural to recall the first method for calculating the critical perturbation used in [1]. The method was based on the direct computation of trajectories for different $K \to K_g$. The criterion of supercriticality of a particular K value was very simple: the transition if only a single trajectory in one of two neighbor integer resonances ($y_r = 0$ *mod* 1) through the destroyed critical curve. With the computers available at that time the minimal $K = 1$ has been reached only which corresponds to the uncertainty $(\Delta K)_{min} = K_{min} - K_g = 0.0284$. This may be compared to the later result $(\Delta K)_{min} = 0.0127$ [10].

Remarkably, the dependence of the average transition time on parameter K was found to be similar to scaling (3):

$$< t > = \frac{A}{(K - K_{cr})^p} \qquad (4)$$

Fitting three unknown parameters gave: $A = 103$, $p = 2.55$, and $K_{cr} = 0.989$. The latter result was rather different from the present value $K_{cr} \approx K_g$, again because of the computation restrictions mentioned above: $K > 1$, $t \leq 10^7$ iterations. Nevertheless, the fitting Eq.(4) provided a less uncertainty

$(\Delta K)_f = K_f - K_g = 0.0174$ as compared to the result from the minimal K. The same is true for data from [10] where $(\Delta K)_f = K_f - K_g = 0.00236$. The latter value was apparently obtained by the direct fitting the relation (3). Fitting in log–log scale provides a much better result: $(\Delta K)_f = K_f - K_g = -0.000128 \pm 0.000288$ that is the remainig uncertainty reduces down to 0.000288.

In both cases the fitted value for the critical perturbation K_{cr} is only true up to a certain confidence probability while the minimal K is an exact result: $K_{cr}^+ = K_{min}$.

In the present paper the studies [1] are continued with much better computers. The main result is farther considerable increasing of the accuracy $(\Delta K)_{cr}$.

To reduce the computation expenses, the transition time was calculated for a number of trajectories N_{tr} started near the unstable fixed point of a half–integer resonance ($y_r = 1/2$ *mod* 1), and then run until each of them crossed over to a neighbor integer resonance.

The minimal K value is determined already by the first trajectory escaped from the half–integer resonance. In this way the minimal uncertainty

$$(\Delta K)_{min} = K_{min} - K_g = 0.00025 \qquad (5)$$

has been achieved with the escape time $t \approx 6.77 \times 10^{11}$ iterations which took about 72 hours of CPU time on ALPHA–4100 computer (see Fig.1).

The average transition time was computed from $N_{tr} = 400$ trajectories for each of 100 values of K in the interval: $0.0035 \leq K - K_g \leq 0.35$. This costed 36 hours of computation. The results are shown in Fig.1. In the whole interval of ΔK the dependence $< t(K) >$ is not exactly a power–law. It becomes so asymptotically for $K \to K_{cr}$ as expected from the theory [12]. For this reason, only few smallest K values of the function $< t(K) >$ were taken for the final fitting which is also shown in Fig.1 by the solid line. It is obtained from the fitting 15 left–most points (just up to the first big fluctuation) in log–log scale, and corresponds to the following parameters in Eq.(4):

$$(\Delta K)_f = 0.000125 \pm 0.000267, \qquad p = 2.959 \pm 0.0771, \qquad A = 33 \pm 8 \quad (6)$$

The fitting relative accuracy $rms = 0.071$ is close to, but somewhat larger than, the standard $rms = 1/\sqrt{N_{tr}} = 0.05$. This is seen from the data of 3 single trajectories in Fig.1, too. Notice also 2 very big deviations for the average over 400 trajectories which nature remains unclear. Interestingly, the relative fitting accuracy of the data [10] is considerably higher: $rms = 0.02$.

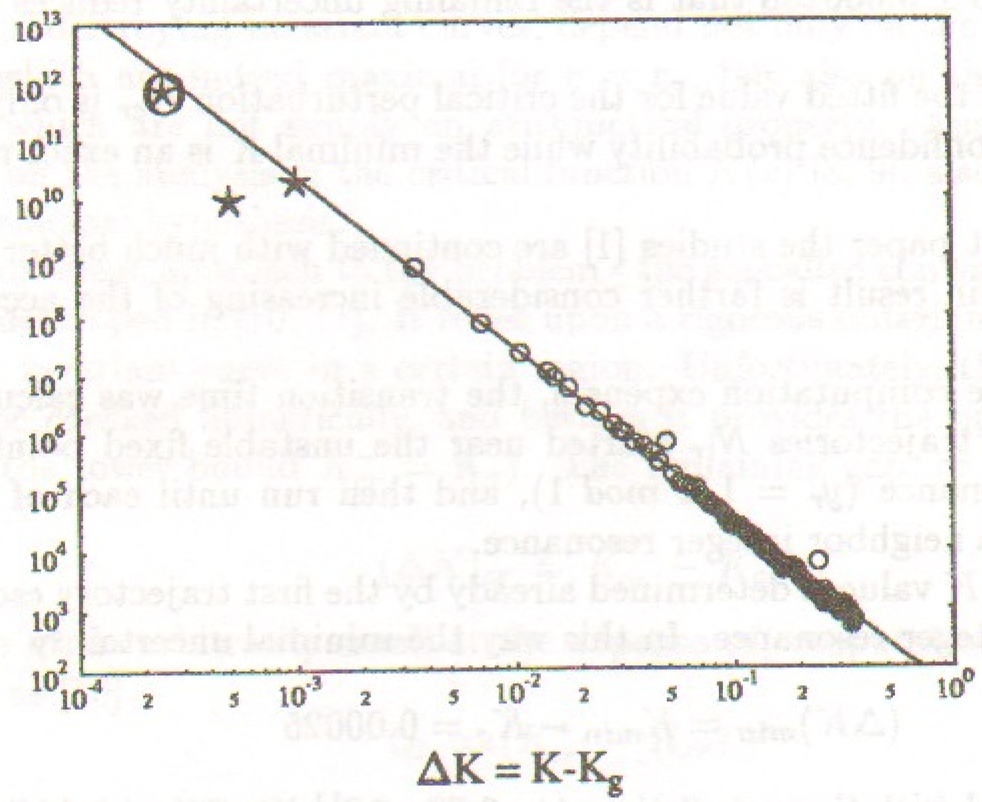

Figure 1: Direct computation of K_{cr} in standard map: average transition time through the destroyed critical curve vs. supercriticality. Circles show numerical results for $N_{tr} = 400$; stars represent 3 single–trajectory runs, including one with the minimal ΔK (5); straight line is relation (4) with parameters (6) fitted from 15 left–most points (circles).

This would require as many as about 5000 trajectories in the present method. However, it does not mean that the computation of the procedure in [10] would be shorter.

The most important parameter in (6) is $(\Delta K)_f$ which is zero within statistical errors. This farther confirms the Greene hypothesis $K_{cr} = K_g$. The exponent p is also equal to the theoretical value $p_{th} = 3.011722$ [12] to the fitting accuracy. The present value of parameter A is much less than in [1] because of a different (shorter) transition between resonances used. The summary of all results is presented in the Table below.

Table. Accuracy of K_{cr} in standard map

$(\Delta K)_{min}$ exact	$(\Delta K)_{fit}$ probable	Reference
2.84×10^{-2}	1.74×10^{-2}	[1]
1.27×10^{-2}	3.36×10^{-3} $\pm 1. \times 10^{-3}$	[10]
	-1.28×10^{-4} $\pm 2.88 \times 10^{-4}$	[10] our fit
2.5×10^{-4}	1.25×10^{-4} $\pm 2.67 \times 10^{-4}$	present paper

A serious difficulty in such a numerical approach to the problem is the computation accuracy. This was mentioned also in [10] but no estimate for the computation errors was given, apparently because of a very complicated numerical procedure. Even in a much simpler method [1], accepted in the present study, the effect of noise turned out to be rather complicated. Special numerical experiments were done to clarify the question. To this end, a random perturbation of amplitude ν was introduced in both equations (1). The results are shown in Fig.2.

Typically, the transition time becomes less than that without noise, and saturates below some critical noise–dependent value of K: $\Delta K \lesssim B(\nu)$. However, in some cases the average transition time considerably grows, as an example in Fig.2 demonstrates, apparently due to a sharp increase of the fluctuations near the crossover from normal (noisefree) dependence of $< t(\Delta K) >$ to the saturation. In turn, these fluctuations are apparently explained by the noise–induced diffusion into some of many small domains of regular motion within the critical structure.

A rough estimate for unknown function $B(\nu)$ can be obtained as follows. The transition time is primarily determined by the width $\delta y \sim (\Delta K)^2$ of the chaotic layer around destroyed critical curve [12, 13, 3] while the diffusion time through this layer $t_0 \sim 1/\Delta K$ [14, 13, 15, 16]. Noise decreases this time down to $t_\nu \sim (\delta y)^2/\nu^2$. Hence, the crossover corresponds to $t_\nu \sim t_0$, whence:

$$B(\nu) \approx A \cdot \nu^b \qquad (7)$$

with $b = 2/5$. Fitting the empirical data in Fig.2 in log–log scale gives: $b \approx 0.39 \pm 0.012$, which is surprisingly close to the theoretical estimate, and $A \approx 0.9716 \pm 0.054$ (Fig.3). The fitting accuracy is also fairly good: the relative $rms = 0.019$. Moreover, below crossover ($\Delta K < B(\nu)$) the width δy as well as the diffusion time depend on ν only, and hence the transition

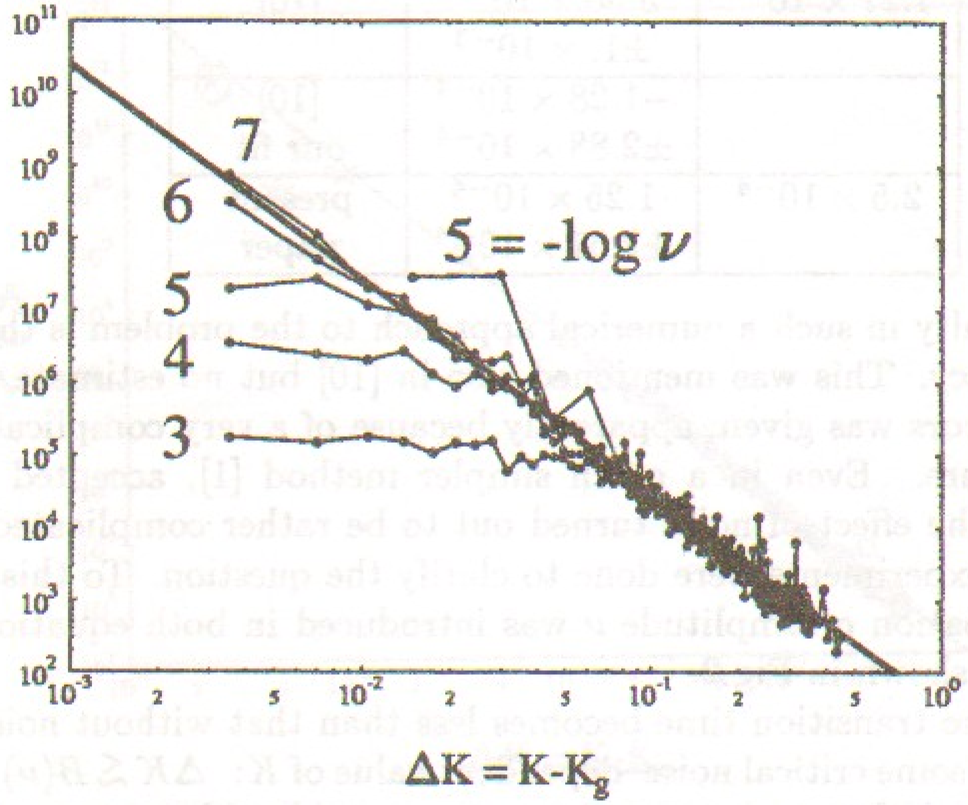

Figure 2: The effect of noise on the supercritical transition time: straight line is the fit of the noisefree computation results (cf. Fig.1); points connected by lines represent the impact of noise with amplitude ν computed for $N_{tr} = 10$; numbers at lines are -log(ν) values (logarithm decimal).

time remains approximately constant for a given ν (Fig.2). In any event, the minimal $(\Delta K)_{min}$ (5), which is the main result of the present study, is well above the expected limitation for the double–precision computation (see Fig.3).

In conclusion, the direct approach *a la* [1] to the problem of the critical perturbation in the standard map does further confirm Greene's hypothesis $K_{cr} = K_g$ with a much better exact upper bound (5): $K_{cr} - K_g < 2.5 \times 10^{-4}$.

Still another recent confirmation of this conjecture (curiously, with roughly the same statistical accuracy (6)) has been inferred from a detailed study of the critical structure at the chaos–chaos border in standard map for $K = K_g$ [16].

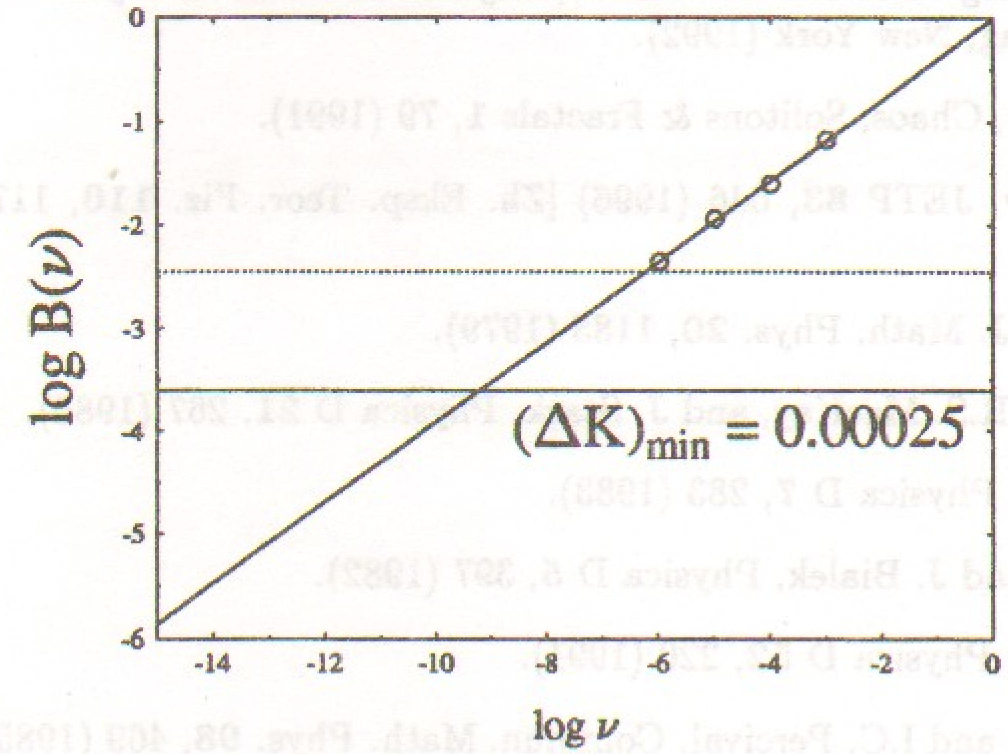

Figure 3: Noise scaling: circles give the empirical crossover values $B(\nu)$ as a function of noise amplitude ν connected and extrapolated by the straight line (7); the upper horizontal dotted line shows the minimal ΔK in computation with noise while the lower line indicates $(\Delta K)_{min}$ (5) achieved in the main double–precision computation (see Fig.1) with the accuracy roughly corresponding to $\log(\nu) \approx -15$; all logarithms are decimal.

Acknowledgements. I am grateful to D.L. Shepelyansky for interesting discussions. This work was partially supported by the Russia Foundation for Fundamental Research, grant 97–01–00865.

References

[1] B.V. Chirikov, Phys. Rep. **52**, 263 (1979).

[2] A.J. Lichtenberg and M.A. Lieberman, *Regular and Chaotic Dynamics*, Springer-Verlag, New York (1992).

[3] B.V. Chirikov, Chaos, Solitons & Fractals **1**, 79 (1991).

[4] B.V. Chirikov, JETP **83**, 646 (1996) [Zh. Eksp. Teor. Fiz. **110**, 1174 (1996)].

[5] J.M. Greene, J. Math. Phys. **20**, 1183 (1979).

[6] J.M. Greene, R.S. MacKay, and J. Stark, Physica D **21**, 267 (1986).

[7] R.S. MacKay, Physica D **7**, 283 (1983).

[8] G. Schmidt, and J. Bialek, Physica D **5**, 397 (1982).

[9] N.W. Murray, Physica D **52**, 220 (1991).

[10] R.S. MacKay, and I.C. Percival, Commun. Math. Phys. **98**, 469 (1985).

[11] J. Stark, ibid. **117**, 177 (1988).

[12] R.S. MacKay, J.D. Meiss, and I.C. Percival, Physica D **13**, 55 (1984).

[13] B.V. Chirikov, Proc. Intern. Conf. on Plasma Physics, Lausanne, **2**, 761 (1984).

[14] B.V. Chirikov, Lect. Notes Phys. **179**, 29 (1983).

[15] S. Ruffo, and D.L. Shepelyansky, Phys. Rev. Lett. **76**, 3300 (1996).

[16] B.V. Chirikov, and D.L. Shepelyansky, Asymptotic Statistics of Poincaré Recurrences in Hamiltonian Systems with Divided Phase Space, preprint Budker INP 98-63, Novosibirsk, 1998; Phys. Rev. Lett. (submitted).

B.V. Chirikov

Critical perturbation in standard map:
A better approximation

Б.В. Чириков

Критическое возмущение в стандартном отображении:
улучшенное приближение

Budker INP 98-63

Ответственный за выпуск А.М. Кудрявцев
Работа поступила 14.10.1998 г.

Сдано в набор 20.10.1998 г.
Подписано в печать 20.10.1998 г.
Формат бумаги 60×90 1/16 Объем 0.8 печ.л., 0.7 уч.-изд.л.
Тираж 130 экз. Бесплатно. Заказ № 75

Обработано на IBM PC и отпечатано на
ротапринте ИЯФ им. Г.И. Будкера СО РАН
Новосибирск, 630090, пр. академика Лаврентьева, 11.