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Budker Institute of Nuclear Physics

G. Casati, B.V. Chirikov, I. Guarneri, F.M. Izrailev

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НОВОСИБИРСК

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G.Casati [a,b], B.V.Chirikov [c], I.Guarneri [a,d], F.M.Izrailev [c]

[a] *Univesita'di Milano, sede di Como, Via Castelnuovo 7, 22100 Como, Italy*

[b] *Instituto Nazionale di Fisica Nucleare, Sezione di Milano, Milano, Italy*

[c] *Budker Istitute of Nuclear Physics, 630090 Novosibirsk, Russia*

[d] *Instituto Nazionale di Fisica Nucleare, sezione di Pavia, 27100 Pavia, Italy*

ABSTRACT

We study the Band Random Matrix model for conservative Hamiltonian systems, originally proposed by Wigner in 1955. On the basis of numerical data we show that both the global structure of eigenstates and the level statistics obey a simple scaling law based on a single scaling parameter.

The first attempt to describe statistical properties of complex quantum systems by means of a random matrix model goes back to Wigner [1]. He introduced a Band Random Matrix (BRM) model to describe conservative systems like atomic nuclei [2]. Specifically, he considered the ensemble of real, infinite, Hamiltonian matrices of the type

$$H_{mn} = \frac{m}{\rho}\delta_{mn} + v_{mn}\,, \qquad v_{mn} = v_{nm} \tag{1}$$

where ρ is the mean level density and the off-diagonal matrix elements are random and statistically independent with $< v_{mn} >= 0$ and $< v_{mn}^2 >= v^2$ for $|m-n| < b$ where b is the band-width while $v_{mn} = 0$ otherwise. In particular, the simplest case of matrix elements with random signs $v_{mn} = \pm v$ was chosen in Ref. [1].

Wigner introduced the weighted level density

$$\rho_W(E;\, m) = \sum_l a_{ml}^2\, \delta(E - E_l) \tag{2}$$

where a_{ml} are components of the eigenfunctions ψ_l of Hamiltonian (1) in a physically significant, for example, unperturbed basis $\{\varphi_m\}$:

$$\psi_l = \sum_m a_{lm}\, \varphi_m \tag{3}$$

and where E_l are eigenvalues corresponding to ψ_l.

The weighted level density $\rho_W(E;\, m)$, termed *strength function* by Wigner, proved to be very useful in studies of quantum statistics, and now is called *local spectral density* (see, e.g., Ref. [3]). It is directly related to the time dependent Green function of a basic state φ_m and characterises the level density of the so-called *operative eigenfunctions* [4] which actually control the dynamics of this initial state.

The analytical evaluation of the density (2) turned out to be extremely difficult. Only in some limit cases Wigner was able to derive an explicit expression for the density

(2). In particular, for $\rho = \infty$ and $b \gg 1$, he obtained the semicircle law

$$\rho_W(E;\, m) = \frac{1}{4\pi v^2 b}\sqrt{8bv^2 - E^2} \tag{4}$$

(a rigorous proof of this law is given in recent papers [5]).

After Wigner's pioneering work the BRM were almost forgotten (curioulsy enough by Wigner himself [6]) apparently because of their mathematical inconvenience, namely, non-invariance with respect to basis rotation. Due to this, attention was paid mainly to full random matrices for which fairly complete mathematical analysis was possible [7].

Nevertheless, full random Hamiltonian matrices can be used to describe only local statistical properties of spectra and were criticized by Dyson for "unphysical" semicircle law. However, a physically meaningful approach to the analysis of global properties of Hamiltonians can be obtained by just going back to the original Wigner model with increasing diagonal elements (1). In this model the semicircle law holds for the weighted level density (1) only, while the total level density is approximately uniform in the semiclassical region. Moreover, in physical applications the interaction of unperturbed states always has a finite range which determines the band structure of Hamiltonian. For this reason, there has been a revival of interest in BRM [8]. Particulary, in Refs. [9, 10] the original Wigner model (1) with increasing diagonal elements has been restored and studied. Another source of interest in BRM is related to solid state physics where band matrices are widely used to describe dynamics of electrons in disordered solids. Here the localization properties of eigenstates are important as well as their relation to the spectrum properties.

In this paper we consider model (1) with gaussian distribution for off-diagonal elements. On the basis of numerical data in Refs. [9, 10] we show that the global structure of eigenfunctions can be described by a simple scaling theory based on a single scaling parameter which has a simple physical meaning. A similar scaling approach accounts very well also for energy level statistics and gives a new insight into the connection between the statistical properties of eigenstates and those of the eigenvalues.

Our starting point is the semicircle law (4) which holds for sufficiently large level density [11–13]. The finite energy width of the semicircle distribution, $\Delta E = 4v\sqrt{2b}$, encompasses the whole spectrum and therefore allows for an estimate of the maximum number $l_\perp$ of unperturbed states that can be coupled by the perturbation:

$$l_\perp \equiv c\rho\,\Delta E = 4c\rho\, v\sqrt{2b} \tag{5}$$

where the numerical factor c (of the order one) depends on how $l_\perp$ is practically measured (see below). The physical meaning of $l_\perp$ is that of a maximal localization length of eigenstates in number of unperturbed levels. We emphasize that $l_\perp$ is determined by the "energy shell" ΔE and makes no reference to any finite matrix size; for this reason $l_\perp$ will be called *transverse localization length* (across the energy shell), see Ref. [14]. The actual localization length l, however, is in general different. It depends on the various parameters in a complicated way [9, 10], except in the limit case $\rho = \infty$, when $l = l_\infty \sim b^2$ [15, 16].

The key point in our approach is that all global properties of eigenfunctions are described by one *localization parameter*:

$$\beta_{loc} = \frac{l}{l_\perp} \tag{6}$$

which is expected to obey a scaling law, i.e. to depend only on the ratio of the two characteristic lengths l_∞ and $l_\perp$:

$$\beta_{loc} = \beta_{loc}(\lambda), \qquad \lambda = \frac{l_\infty}{l_\perp} = a\,\frac{b^{3/2}}{\rho\, v} \tag{7}$$

with some numerical factor $a \sim 1$. The scaling parameter λ may also be called *ergodicity parameter*, because when it is large, the localization length approaches its maximal value $l_\perp$, which means that the eigenfunctions become ergodic, i.e., completely delocalized within the energy shell. In the opposite case, when $l < l_\perp$, we speak of *logitudinal localization* (along the layer) [14].

The scaling parameter λ, which in our approach naturally appears as the ratio of two characteristic lengths, was introduced in Refs. [9, 10, 15] where *finite* matrices of size N were considered. In that case, N is another characteristic length, and therefore, besides λ, one more scaling parameter appears:

$$\lambda_N = \gamma\,\frac{b^2}{N} \tag{8}$$

The statistical properties were conjectured to depend on both parameters λ and λ_N ($\beta_{loc} = \beta_{loc}(\lambda, \lambda_N)$) [9]. This was confirmed by numerical experiments [9, 10, 17], both for spectral statistics and for localization properties.

For homogeneous ($\rho = \infty$) BRM the parameter λ_N alone is sufficient, and the scaling law was numerically found to be (with $\gamma \approx 1.4$):

$$\beta_{loc}(\lambda_N) \approx \frac{\lambda_N}{1 + \lambda_N} \tag{9}$$

for $\lambda_N \lesssim 14$ [16, 18]. Recently, an analytical proof of this scaling law was given in Ref. [21].

However, we are here interested in the case of infinite matrices. Though in numerical studies the matrices are always of finite size N, the data in Refs. [9, 10] pertain to large matrices for which $\lambda \gg \lambda_N$. According to numerical evidence [9, 10], in this region only the parameter λ is important, and the finite (but large) size of the matrix is not relevant.

In order to study the scaling (7) we use numerical data [9, 10] for the dependence of localization length on the model parameters,

$$l(x) = b^2 f(x) \tag{10}$$

where $x = \lambda a^{-1} = b^{3/2}/(\rho v)$ and function $f(x)$ was numerically found in Refs. [9, 10].

In this formulation, the asymptotic behaviour of $\beta_{loc}(\lambda)$ as $\lambda \to \infty$ is determined by the asymptotics of $x f(x)$ for $x \to \infty$. In Ref. [15] it was argued that $x f(x)$ tends to some nonzero limit c_0 as $x \to \infty$, asymptotically for $1 \ll b \ll N$. The definition of $l(x)$ which was used in actual data, was that of "entropy localization length", namely,

$$l = N \exp(< H > - H_{GOE}) \tag{11}$$

where H is the "entropy" of an eigenstate u_n:

$$H = -\sum_n |u_n|^2 \ln |u_n|^2 \tag{12}$$

and $< H >$ is an average over all eigenvectors of an ensemble of random matrices with the same N, ρ, b, v. In the expression (11) the normalization is used in such a way that for the limit case of full random matrices of size N the localization length l is equal to N [18]. For this, the factor $\exp(-H_{GOE})$ is introduced which is the average entropy of eigenstates taken from the Gaussian Ortogonal Ensemble (GOE). The definition (11–12) has the meaning of an effective number of unperturbed eigenstates significantly covered by a given eigenvector. From the data of Ref. [9] and from additional ones kindly provided by M.Feingold, on increasing x at fixed $b = 3 - 9$, the product $x f(x)$ was found to increase towards a maximum value in the range $4.4 - 5.5$, and then to decrease approximately as $1/x$; moreover, in this region, the scaling (7) is violated. This behaviour has been related to perturbation theory [11, 13]. Being mainly interested in

non-perturbative effects, we restricted ourselves to the range of x values before the maximum of $x f(x)$. This range is approximately defined [11] by the condition $x \leq x_b$ with

$$\frac{b}{x_b} = \frac{\rho v}{\sqrt{b}} \sim 0.3 \tag{13}$$

The quantity b/x is a sort of perturbation parameter, corresponding to the ratio of the rms perturbation ($\sim v\sqrt{b}$) to the full detuning ($\sim b/\rho$). Another interpretation of border (13) is the ratio $l_\perp/b$ of the maximal (transverse) localization length to the number of directly coupled unperturbed states.

As b increases, the border x_b also increases, and the maximum of $x f(x)$ becomes broad and flat; this justifies our restriction $x \leq x_b$ and the determination of c_0 as a fitting parameter. The direct calculation of the constant c_0 requires new data for larger b. For the largest available $b = 9$, the mean value of $x f(x)$ on the plateau (which includes only 7 points) is 5.42, while the value of c_0 obtained from the fitting (see Fig. 1) is $c_0 = 5.29$.

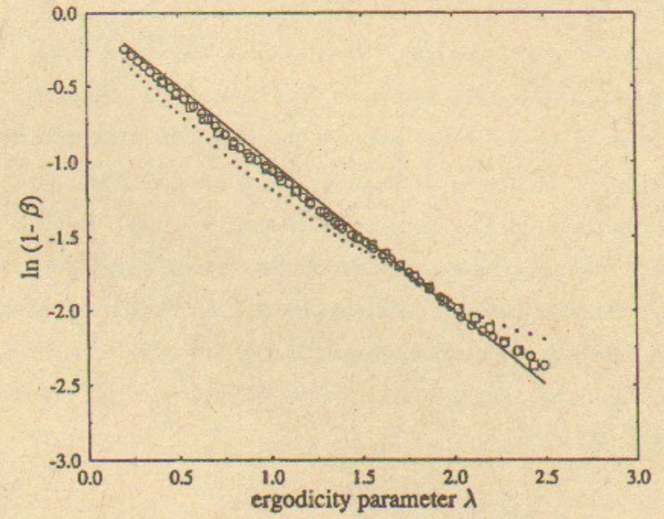

Fig. 1. Dependence of the localization (squares) and repulsion (dots for q and circles for β (16)) parameters on the ergodicity parameter λ as compared to the simple exponential scaling (14) (solid line): the rms deviation for $\beta_{loc} - \beta$ is 0.7%.

A detailed processing of the data in the region $x < x_b$ yields evidence of a simple scaling (Fig. 1):

$$\beta_{loc}(\lambda) = \frac{x f(x)}{c_0} \approx 1 - \exp(-\lambda) \equiv \beta_0(\lambda) \qquad (14)$$

where β_0 is defined by the last equality. To suppress fluctuations in numerical data for large λ we used the so-called moving window averaging over 6 neighbouring points. The least square fit gives $c_0 = 5.29\,, a = 0.216$ with rms deviation of 6 %. We would like to stress that the scaling (14) is quite different from (9) found for finite homogeneous BRM.

Although the agreement between numerical data and the scaling law (14) shown in Fig.1 is overall quite good, there is also a small systematic deviation. This deviation $\beta_{loc} - \beta_0$ can be substantially reduced by the slight change in the scaling (14), namely

$$\beta_0(\lambda) \to \beta_1(\lambda) = 1 - \exp(-\lambda + \mu\lambda^2 - \nu) \qquad (15)$$

with $\mu = 0.039$, $\nu = 0.055$ and a close value of $a = 0.226$ ($c_0 = 5.29$ as before). Using the latter parameter values, the numerical factor $\gamma \approx ac_0 \approx 1.2$ is also close to the previously found one in different models [16]. Factor $c \approx 1.07$ in Eq. (5) corresponds to a particular definition of localization length used in all the above mentioned numerical experiments, namely, the entropy localization length [18].

Another goal of this paper was to investigate the relation of the above discussed global properties of the eigenfunctions to the statistical properties of the energy spectrum. The most widely used quantity for the latter is the repulsion parameter for couples of neighbouring levels [7]. To determine this parameter one needs to compare numerical data with some expression for the distribution of level spacings. In Ref.[10] this was done for the model (1) by using the so-called Brody distribution, which depends on one fitting parameter q. We used instead a different theoretical distribution, which appears more physical in the analysis of intermediate statistics produced by localization effects [18, 19]. This distribution, too, depends on one *spectral parameter* β, which is approximately related to q [19]:

$$\beta = 0.654\,q + 0.411\,q^2 \qquad (16)$$

This relation was obtained by a least square fitting of one distribution to the other. In previous studies [18, 20] of dynamical models it was conjectured and numerically supported that the localization parameter β_{loc} is close to the repulsion parameter β.

In the present case, this conjecture is fully supported by numerical data, see Fig. 1. We emphasize that in spite of some statistically significant deviations of both β and β_{loc} from the simple exponential scaling (14), the difference $\beta_{loc} - \beta$ remains well within statistical fluctuations in the whole range of available data for $q(\lambda)$. The rms fluctuations of $(\beta_{loc} - \beta)/(1 - \beta)$ are about 2% only. This in our opinion clearly indicates that parameter β is much more suited than the Brody parameter for the description of statistical properties of quantum chaos.

It would be very interesting to extend numerical experiments [9, 10] on larger values of ergodicity patameter λ in order to follow the transition to the unperturbed system as represented by the diagonal matrix elements. Also, it is not completely clear whether the present results remain unchanged with the different, particulary Poissonian statistics of the diagonal matrix elements in the model (1). Most likely, they do but a direct check would be certainly desirable.

In conclusion, we have provided evidence for the new scaling law which holds for BRM of the form (1); besides that, we have shown that for this model the spectral parameter β and the localization parameter β_{loc} are surprisingly close. The latter striking result is still waiting for a theoretical explanation.

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