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MULTIBUNCH RESISTIVE WALL INSTABILITIES OF AN INTENSE ELECTRON BEAM IN STORAGE RINGS

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НОВОСИБИРСК

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ABSTRACT

A coherent interaction of an electron beam with a surrounding structure can influence essentially on a beam dynamics. We have studied the resistive instability of the longitudinal and transverse motion of a multibunch beam in a storage ring due to the finite conductivity of chamber walls.

It is known that the positive and negative sign of a single bunch transverse oscillations growth rate depends on the fractional portion of ν_x- betatron oscillations tune. Unfortunately, in the multibunch regime there always exist oscillations modes with positive growth rates.

As for longitudinal oscillations of a single bunch their growth rate is determined by a differences of the real parts of the impedance at the upper and lower side frequencies for the harmonics of the particle revolution frequency ω_0, these differences being of the order $\Omega/\omega_0 \ll 1$(Ω - synchrotronous frequency) because the resistive chamber impedance smoothly depends on the frequency (as $\sqrt{\omega}$). In the multibunch case the expression for the growth rate includes these differences at the upper and the lower side-frequencies not for the same, but for different harmonics of the revolution frequency. The small factor Ω/ω disappears.

That is why both for longitudinal and for transverse oscillations of the multibunch beam there arises the problem to find their maximum possible growth rates and to compare them with radiation damping.

We have shown that the longitudinal resistive wall instability of the multibunch beam has negligibly small growth rate compare to the single bunch one. But the growth rates of the transverse resistive wall instability can exceed radiation damping. A possibility to decrease the transverse instability growth rate in the case of the beam with some empty buckets has been analyzed. It was shown that it is not a way to cure this instability.

INTRODUCTION

The influence of the coherent interaction of an electron beam with a rather large intensity with the surrounding structure on a beam dynamic can be essential. So, it is interesting to study the ultrarelativistic beam's interaction with the electromagnetic field induced by this beam in the vacuum chamber with the walls of finite conductivity [1], [2].

It is known that the sign of the growth rate of a single bunch transverse oscillations is determined by the fractional portion of the betatron oscillation tune ν_x. Unlike, in the multibunch regime there always exist oscillations modes with positive growth rates.

As for longitudinal oscillations of a single bunch their growth rate is determined by a differences of the real parts of the chamber impedance at the upper and lower side-frequencies for the harmonics of the particle revolution frequency ω_o [3]:

$$\sigma = A\sum_m m(R_m^+ - R_m^-)\ , \text{ where } \ R_m^{\pm} = \mathrm{Re}(Z_o(m\omega \pm\Omega)),$$

Ω - synchrotron oscillation frequency.

As the resistive impedance has a smooth dependence on the frequency (as $\sqrt{\omega}$), these differences are of order $\Omega\ \omega_o \ll 1$ (fig. 1a).

In the multibunch case the expression for the growth rate includes the differences of the real parts of the impedance at the upper and lower side-frequencies not for the same, but for different harmonics of the revolution frequency: the small factor Ω/ω_o disappears.

That's why both for longitudinal and for transverse oscillations of the multibunch beam there arises the problem to determine their maximum growth rates and to compare them with radiation damping.

The instabilities has been studied with the method developed in [3]: firstly we determine a field induced by a beam due to its coherent oscillations; secondly we write the equations of motion of the test particles in this field; and then identifying the test particles with ones inducing the field we analyze the stability of the beam motion. We suppose here for the sake of simplicity azimuthally uniform focusing and represent each bunch as one macroparticle.

1. The Electromagnetic Field Induced by a Beam in the Waveguide with the Walls of Finite Conductivity

The Laplace transform of the electromagnetic field induced by a current in the waveguide must satisfy the Maxwell equations (1) and the Leontowitch boundary conditions (2):

$$\mathbf{rotE} = -s\mu_o \mathbf{H}$$
$$\mathbf{rotH} = s\varepsilon_o \mathbf{E}+\mathbf{j} \qquad (1)$$
$$\mathbf{E}\times\mathbf{n} = \xi(\mathbf{H}\times\mathbf{n})\times\mathbf{n}, \qquad (2)$$

where $\mathbf{n}$ is a unit vector normal to the surface; s is a Laplace variable; $\xi=\sqrt{s\mu/\sigma} = Z_o(\delta_o/R)\sqrt{s/2\omega_o}$ the metal surface impedance; σ and μ – its conductivity and magnetic permeability; $\delta_o = \sqrt{\dfrac{2}{\omega_o\sigma\mu}}$ – a thickness of the skin at the revolution frequency; $Z_o = \mu c = 120\pi$ Ohm – free space impedance; R is a radius of a storage ring.

As the metal conductivity is large we can expand the field over the powers of ξ and drop the items of order more then one:

$$\mathbf{E} = \mathbf{E}^o + \xi\mathbf{E}^1 = \mathbf{E}^o + \mathbf{E}_{add},$$
$$\qquad (3)$$
$$\mathbf{H} = \mathbf{H}^o + \xi\mathbf{H}^1 = \mathbf{H}^o + \mathbf{H}_{add}.$$

The fields $\mathbf{E}^o$ and $\mathbf{H}^o$ satisfy the Maxwell equations (1) and zero boundary conditions. The additional fields $\mathbf{E}_{add}$ and $\mathbf{H}_{add}$ arising due to the finite conductivity satisfy the Maxwell equations with the zero right-hand side and boundary conditions (4) following from (2) and (3):

$$\mathbf{E}_{add}\times\mathbf{n} = \xi\mathbf{H}^o. \qquad (4)$$

According to [4] we can write a magnetic field $\mathbf{H}^o$ in an ideal waveguide as a sum over its eigenfunctions:

$$\mathbf{H}^o(x,y,l,s) =$$
$$= -\sum_k \frac{1}{2D_k}\Big\{\mathbf{H}_k(x,y,s)e^{-\gamma_k l}\int_{-\infty}^{l} e^{\gamma_k l'}dl'\int_S \mathbf{E}_{-k}(x,y,s)\mathbf{j}(x',y',l',s)dx'dy' +$$
$$+\mathbf{H}_{-k}(x,y,s)e^{\gamma_k l}\int_{l}^{+\infty} e^{-\gamma_k l'}dl'\int_S \mathbf{E}_k(x,y,s)\mathbf{j}(x',y',l',s)dx'dy'\Big\}. \qquad (5)$$

Here $\mathbf{E}_k$, $\mathbf{H}_k$ are the waveguide eigenfunctions for the waves propagating in the positive (for k>0) or negative (for k<0) l – direction; γ_k is a propagation number for these modes; $D_k = \int_S (\mathbf{E}_k \times \mathbf{H}_k)_l dS$; S is the waveguide cross section. The waveguide eigenfunc- tions can be expressed via the membrane functions φ_k x, y), $\psi_k(x, y)$ (Appendix 1).

Expanding the beam current over the azimuthal harmonics according to [3] and integrating in (5) over l′, we get:

$$H^o(x,y,l,s) = -\sum_{m,k} \frac{e^{iml/R}}{4\pi R D_k}\left\{ H_k(x, y, s)\frac{I_{km}(s+im\omega_o)}{\gamma_k+im/R} + H_{-k}(x, y, s)\frac{I_{-km}(s+im\omega_o)}{\gamma_k-im/R}\right\},$$

where

$$I_{km}(s+im\omega_o) = 2\pi R\int_S E_{-k}(x',y',s)\,j_m(x',y',s+im\omega_o)\,dx'dy', \quad (6)$$

$$j_m(x, y, s) = \frac{1}{2\pi R}\int_{-\pi R}^{\pi R} j(x, y, z, s)e^{-imz/R}dz,$$

$$z = l - vt, \quad v = \omega_o R.$$

The additional fields E_{add} and H_{add} we search in a form

$$E_{add} = \sum_{m=-\infty}^{\infty} E_m(x, y, s)e^{iml/R}, \quad H_{add} = \sum_{m=-\infty}^{\infty} H_m(x, y, s)\,e^{iml/R}. \quad (7)$$

The fields E_{add} and H_{add} satisfying the homogeneous Maxwell equations can be expressed via the membrane functions $\Phi_m(x, y, s)$ and $\Psi_m(x, y, s)$:

$$E_m = -\mathrm{grad}\Phi_m + \frac{(m/R)^2+s^2\mu_o\varepsilon_o}{im/R}\Phi_m l_o + l_o\times\mathrm{grad}\Psi_m,$$

$$H_m = \frac{im/R}{s\mu_o}\mathrm{grad}\Psi_m - \frac{(m/R)^2+s^2\mu_o\varepsilon_o}{s\mu_o}\Psi_m l_o + \frac{s\varepsilon_o}{im/R}l_o\times\mathrm{grad}\Phi_m, \quad (8)$$

where Φ_m and Ψ_m satisfy the equations:

$$\Delta\Phi_m - ((m/R)^2+s^2\mu_o\varepsilon_o)\Phi_m = 0; \quad \Delta\Psi_m - ((m/R)^2+s^2\mu_o\varepsilon_o)\Psi_m = 0, \quad (9)$$

and the boundary conditions are get by putting (6), (7) and (8) into (4):

$$\Phi_m\Big|_C = -\frac{\xi/2}{2\pi R}\,\frac{im/R}{(m/R)^2+s^2\mu_o\varepsilon_o}\times\left\{\sum_k\frac{\partial\phi_k}{\partial n}\left(\frac{I_{km}}{\gamma_k+im/R}-\frac{I_{-km}}{\gamma_k-im/R}\right)-\sum_k\frac{\partial\psi_k}{\partial\tau}\,\frac{2im/R}{\gamma_k'^2+(m/R)^2}I'_{km}\right\}, \quad (10)$$

$$\frac{\partial\Psi}{\partial n}\Big|_C = -\frac{\xi/2}{2\pi R}\,\frac{im/R}{(m/R)^2+s^2\mu_o\varepsilon_o}\times\left\{\sum_k\frac{\partial}{\partial\tau}\left(\frac{\partial\phi_k}{\partial n}\right)\left(\frac{I_{km}}{\gamma_k+im/R}-\frac{I_{-km}}{\gamma_k-im/R}\right)-\sum_k\frac{\partial^2\psi_k}{\partial\tau^2}\,\frac{2im/R}{\gamma_k'^2+(m/R)^2}I'_{km}\right\}-\frac{\xi}{4\pi R}\sum_k\psi_k\,\frac{2g_k'^2}{\gamma_k'^2+(m/R)^2}I'_{km}. \quad (11)$$

Using the equations (9) and the boundary conditions (10) and (11) we can find the functions Φ_m and Ψ_m and thus the additional fields arising due to the waveguide walls finite conductivity.

The next calculations were made for the waveguide with a rectangular cross-section a x b, normalized eigenfunctions having a form:

$$\phi_{kr} = \frac{2}{g_{kr}\sqrt{ab}} \sin\left(\frac{k\pi y}{a} - \frac{k\pi}{2}\right) \sin\left(\frac{r\pi x}{b} - \frac{r\pi}{2}\right),$$

$$\psi_{kr} = \frac{2}{g_{kr}\sqrt{ab}} \cos\left(\frac{k\pi y}{a} - \frac{k\pi}{2}\right) \cos\left(\frac{r\pi x}{b} - \frac{r\pi}{2}\right), \tag{12}$$

where

$$g_{kr}^2 = \left(\frac{k\pi}{a}\right)^2 + \left(\frac{r\pi}{b}\right)^2 \tag{13}$$

Solving equations (9) we can write the functions Φ_m and Ψ_m in a form:

$$\Phi_m = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi y}{a} - \frac{k\pi}{2}\right) (C'_k \mathrm{sh}(h_{mkx} x) + D'_k \mathrm{ch}(h_{mkx} x)) +$$
$$+ \sum_{r=1}^{\infty} \sin\left(\frac{r\pi x}{b} - \frac{r\pi}{2}\right) (C''_r \mathrm{sh}(h_{mry} y) + D''_r \mathrm{ch}(h_{mry} y)), \tag{14}$$

$$\Psi_m = \sum_{k=1}^{\infty} \cos\left(\frac{k\pi y}{a} - \frac{k\pi}{2}\right) (M'_k \mathrm{sh}(h_{mkx} x) + N'_k \mathrm{ch}(h_{mkx} x)) +$$
$$+ \sum_{r=1}^{\infty} \cos\left(\frac{r\pi x}{b} - \frac{r\pi}{2}\right) (M''_r \mathrm{sh}(h_{mry} y) + N''_r \mathrm{ch}(h_{mry} y)), \tag{15}$$

$$h_{mkx} = \sqrt{\left(\frac{k\pi}{a}\right)^2 + \left(\frac{m}{R}\right)^2 + s^2\mu_o\varepsilon_o},$$

$$h_{mry} = \sqrt{\left(\frac{r\pi}{b}\right)^2 + \left(\frac{m}{R}\right)^2 + s^2\mu_o\varepsilon_o}. \tag{16}$$

The coefficients in (14), (15) can be found from the boundconditions (10), (11) (Appendix 2).

2. The Motion Equations in the Action - Phase Variables

By studying the stability of the beam oscillations it is useful to turn to the slowly changing variables phase and action [3]:

$$z = \sqrt{2J_z/(M\Omega)}\,\sin(\psi_z), \quad \dot{z} = \Omega\sqrt{2J_z/(M\Omega)}\,\cos(\psi_z);$$

$$x = \sqrt{2J_x/(m_s\Omega_x)}\,\sin(\psi_x), \quad \dot{x} = \Omega_x\sqrt{2J_x/(m_s\Omega_x)}\,\cos(\psi_x);$$

where $M = \frac{m_s}{\gamma_s^{-2} - \alpha}$, m_s and γ_s are a mass and relativistic factor of the equilibrium particle, α is a momentum compaction factor; Ω and Ω_x are the frequencies of the longitudinal (synchrotron) and transverse (betatron) oscillations.

Further we will drop the subscript 'add', all mentioned field components concern to the additional field.

The equations of motion in the new variables have a form [3]:

$$\dot{J}_z = 2\sigma_z \dot{J}_z = e\,\overline{\frac{\partial z}{\partial \psi_z} E_z}, \quad \dot{\psi}_z = \Omega + e\,\overline{\frac{\partial z}{\partial J_z} E_z}. \tag{17}$$

$$\dot{J}_x = 2\sigma_x \dot{J}_x = e\,\overline{\frac{\partial x}{\partial \psi_x} (E_x - vB_y)}, \quad \dot{\psi}_x = \Omega_x + e\,\overline{\frac{\partial x}{\partial J_x} (E_x - vB_y)}. \tag{18}$$

We take here the longitudinal field E_z on the equilibrium orbit and the transverse fields E_x and B_y in a paraxial approach.

A line over the r.h.s. of these equations means that these expressions must be averaged over the time much more than the period of the fast oscillations [3].

For the case of an symmetrical beam we need only first equations from (17) and (18) because the phase shifts can be determined from the symmetry condition.

3. A Longitudinal Instability

By calculating the longitudinal field on the equilibrium orbit of the waveguide we must take into account that the transverse components of a beam current is much less than the longitudinal one, and that a longitudinal field changes its sign while changing the direction of the propagation; that is

$$I'_{krm}=0,\quad I_{krm}=-I_{-krm}, \qquad (19)$$

and the electric field on the axis in the laboratory reference system has a form:

$$E_l(0,0,l,s)=\frac{1}{2\pi R\sqrt{ab}}\sum_{m=-\infty}^{+\infty}\xi\, e^{iml/R}\sum_{k,r=1}^{+\infty}\frac{2\gamma_{kr}}{g_{kr}}\frac{I_{krm}(s+im\omega_o)}{\gamma_{kr}^2+\left(\frac{m}{R}\right)^2}\times$$

$$\times\left\{\frac{\frac{r\pi}{b}\sin\frac{k\pi}{2}\sin^2\frac{r\pi}{2}}{ch\left(\frac{h_{mkx}b}{2}\right)}+\frac{\frac{k\pi}{a}\sin\frac{r\pi}{2}\sin^2\frac{k\pi}{2}}{ch\left(\frac{h_{mry}a}{2}\right)}\right\} \qquad (20)$$

For the further calculations it is convenient to turn formally to the reference system of the equilibrium particle (i.e. from the longitudinal variable l to $z=l-\omega_o Rt$ in the inverse Laplace transform of (20)), and after this substitution a Laplace transform of the longitudinal electric field can be written as

$$E_z(0,0,z,s)=\frac{1}{2\pi R\sqrt{ab}}\sum_{m=-\infty}^{+\infty}\xi\, e^{imz/R}\sum_{k,r=1}^{+\infty}\frac{2\gamma_{kr}}{g_{kr}}\frac{I_{krm}(s)}{\gamma_{kr}^2+\left(\frac{m}{R}\right)^2}\times$$

$$\times\left\{\frac{\frac{r\pi}{b}\sin\frac{k\pi}{2}\sin^2\frac{r\pi}{2}}{ch\left(\frac{h_{mkx}b}{2}\right)}+\frac{\frac{k\pi}{a}\sin\frac{r\pi}{2}\sin^2\frac{k\pi}{2}}{ch\left(\frac{h_{mry}a}{2}\right)}\right\}, \qquad (21)$$

where

$$I_{krm}(s)=2\pi R\int_S E_{-kr}(x',y',s-im\omega_o)\,j_m(x',y',s)\,dx'dy',$$

and a replacement $s\longrightarrow s-im\omega_o$ was made in all variables depending on s (ξ, γ_{kr}, h_{mkx}, h_{mry}).

A beam consisting of n_o equal bunches symmetrically placed along the storage ring has n_o symmetrical modes of longitudinal oscillations, a current of a mode with a number k being

$$j_z=eN_1 v\delta(x)\,\delta(y)\sum_{n=1}^{n_o}\times$$

$$\times\,\delta\left\{z-\frac{2\pi}{n_o}(n-1)R-z\sin\left[\Omega t+\frac{2\pi}{n_o}(n-1)(k-1)\right]\right\}. \qquad (22)$$

We can calculate I_{krm} for this current distribution and afterinserting the result into the expression for the electric field and averaging over the time the r.h.s. of (18) [3] we get in the linear in J_z approach:

$$\sigma_z^k=\frac{\dot{J}_z}{2J_z}=\frac{I_o\Omega}{2qV}\sum_{p=-\infty}^{+\infty}(pn_o+k)R^+_{pn_o+k}=\frac{I_o\Omega}{2qV}\sum_{p=0}^{+\infty}\left(m_1R^+_{m_1}-m_2R^-_{m_2}\right), \qquad (23)$$

where q is a harmonic number; V is the accelerating voltage; $m_1=pn_o+k$; $m_2=pn_o+n_o-k$; I_o is the full current of a beam, $I_o=eN_1vn_o=eNv$, $R^{\mp}_m=\mathrm{Re}\left[Z(-im\omega_o\pm i\Omega)\right]$,

$$Z(-im\omega_o\mp i\Omega)=\frac{8\pi R\xi}{ab}\sum_{k,r=1}^{+\infty}\frac{1}{\gamma_{kr}^2+\left(\frac{m}{R}\right)^2}\left\{\frac{\frac{r\pi}{b}\sin\frac{r\pi}{2}}{ch\left(\frac{h_{mkx}b}{2}\right)}+\frac{\frac{k\pi}{a}\sin\frac{k\pi}{2}}{ch\left(\frac{h_{mry}a}{2}\right)}\right\}, \qquad (24)$$

where the argument of the functions depending on s must be $-im\omega_o\mp i\Omega$. In the quazistatic approach

$$h_{mkx}\approx\frac{k\pi}{a},\quad h_{mry}\approx\frac{r\pi}{b},\quad \gamma_{kr}^2+\left(\frac{m}{R}\right)^2\approx g_{kr}^2=\left(\frac{k\pi}{a}\right)^2+\left(\frac{r\pi}{b}\right)^2;$$

$$\mathrm{Re}\left[\xi(-im\omega_o\pm i\Omega)\right]\approx\delta_o\,\omega_o\,\mu\,\frac{\sqrt{m}}{2}\left(1\pm\frac{\nu_z}{2m}\right),\quad \nu_z=\Omega\ \omega_o .$$

In the case of a single bunch with a current I_o a difference $(R^+_m - R^-_m)$ has an order of $\nu_z \ll 1$ (fig.1a) and a single bunch growth rate can be written as

$$\sigma_{zo} = 2\pi\Omega \frac{I_o Z_o}{qV} \frac{\delta_o}{\sqrt{ab}} \nu_z F_z\left(\frac{b}{a}\right) \sum_{m=1}^{\infty} \sqrt{m} , \qquad (25)$$

where

$$F_z(t) = \frac{1}{4} \sum_{k-odd} \left[\frac{\sqrt{t}}{ch^2(k\pi t)} + \frac{\sqrt{1/t}}{ch^2(k\pi/t)} \right] \text{ (see fig. 2a) } .$$

We must take into account a nonzero length of a bunch in order to get a convergent sum instead of (25). For a gaussian bunch with a r.m.s. length Δz the m-th current harmonic must be multiplied on $\exp(-m^2\Delta z^2/2R^2)$, and after calculating the average field which acts on such "long" macroparticle this factor appears ones more, and we get

$$\sigma_{zo} = 2\pi\Omega \frac{I_o Z_o}{qV} \frac{\delta_o}{\sqrt{ab}} \nu_z F_z\left(\frac{b}{a}\right) \sum_{m=1}^{\infty} \sqrt{m} \exp(-m^2\Delta z^2/R^2) \approx$$

$$\approx 2\pi\Omega \frac{I_o Z_o}{qV} \frac{\delta_o}{\sqrt{ab}} \nu_z F_z\left(\frac{b}{a}\right) \frac{2}{3} m_o^{3/2} , \quad m_o = \frac{R}{\Delta_z} . \qquad (26)$$

For the numerical calculations we take the proposed parameters of VEPP-5:

$R=100$ m,	$\omega_o=3\cdot10^6$ s^{-1},
$\Delta_z=1$ cm,	$\nu_z=\Omega/\omega_o=0.017$, $\nu_x=\Omega_x/\omega_o=26$,
$a=5$ cm,	$\gamma_s=13\cdot10^3$,
$b=3$ cm,	$V=6\cdot10^6$ V ,
$\lambda_{RF}=60$ cm, $q=2000$	$I_o=1$ A ,
$n_o=150$,	$\rho=1/\sigma=1.72\cdot10^{-8}$ Ohm·m (for cooper),
$\xi_{synch}=400$ s^{-1},	$\delta_o = 10^{-4}$ m ,
$F_z(\frac{b}{a})\approx0.1$,	i.e. $\sigma_{zo} = 0.08$ s^{-1}.

The expression for σ_{zo} includes two small factors, δ_o/R and ν_z. In the case of a multibunch beam a factor ν_z disappears because the sum (23) contains the terms of opposite sign not with equal, but with different numbers m (fig. 1b).

Estimating a tail of the sum (23) up from the item with a number (p_o+1) as an integral we can get the next approach for the growth rates of the multibunch beam longitudinal motion:

$$\sigma^k_z = \sigma_{zo}\left(\frac{1}{n} + S_z(f)\, \frac{2}{3} (\Delta_z n_o/R)^{3/2} / \nu_z \right) ,$$

where

$$f = k/n_o, \quad k = 1,\dots,n_o, \qquad (27)$$

$$S_{p_o}(f) = \sum_{p=0}^{p_o} \left\{ (p+f)^{3/2} - (p+1-f)^{3/2} \right\} + (1-2f)\cdot(p_o+1)^{3/2} .$$

Note that for any p_o $S(0)=S(0.5)=S(1)=0$. Even the zero approach $(p_o = 0)$ gives a rather precise result. Function S_o has extrema in the points $f_e = \frac{1}{2} \pm \frac{2\sqrt{2}}{9}$, $S_o(f_e) \approx \pm 0.03$ (fig. 3a).

A maximum growth rate appears to be much less than the growth rate of the single bunch with the same current I_o:

$$\sigma_{z\,max} = 0.005\cdot\sigma_{zo} = 4\cdot10^{-4}\,s^{-1}.$$

Thus, in spite of absence of the small factor ν_z the effect of dropping the main part of items in the sum over m prevails and the growth rate is still negligibly small.

4. The Transverse Instability

The transverse Lorentz force $E_x - vB_y$ can be expressed via the membrane functions as

$$(E_x - vB_y)_m = -\left(1+\frac{v\mu_o \varepsilon_o s}{im/R}\right)\frac{\partial\Phi_m}{\partial x}\bigg|_{\substack{x=0\\y=0}} - \left(1+\frac{v\,\frac{im}{R}}{s}\right)\frac{\partial\Psi_m}{\partial y}\bigg|_{\substack{x=0\\y=0}}, \quad (28)$$

When turning to the equilibrium particle system of reference the argument of all functions depending on s (except the currents I_{krm} and I'_{krm}) must be replaced on $s-im\omega_o$. That gives for the ultrarelativistic particle:

$$(E_x - vB_y)_m = -\frac{s}{im\omega_o}\frac{\partial\Phi_m}{\partial x}\bigg|_{\substack{x=0\\y=0}} - \frac{s}{s-im\omega_o}\frac{\partial\Psi_m}{\partial y}\bigg|_{\substack{x=0\\y=0}}. \quad (29)$$

The transverse currents can be neglected in comparison with the longitudinal ones. Terms with I'_{krm} disappear as they are calculated for the modes with the zero longitudinal field, and when calculating I_{krm} we must use the paraxial approach:

$$E_{-kr,1} \approx -\frac{g^2_{kr}}{\gamma_{kr}}\left(\phi_{kr}\bigg|_{\substack{x=0\\y=0}} + x\frac{\partial\phi_{kr}}{\partial x}\bigg|_{\substack{x=0\\y=0}}\right). \quad (30)$$

The first item can be dropped when calculating I_{krm} as it becomes zero after averaging over the time in the equation (18). As a result the field excited at the point with the coordinates (0,0,z) by a macroparticle with a current I_o and coordinates $(x_o,0,z_o)$ can be written as

$$(E_x - vB_y)_m\Big|_z = -\frac{4\omega_o R I_o}{ab}\sum_m \frac{\xi}{s-im\omega_o} e^{imz/R}\,\mathscr{L}\left(x_o e^{-imz_o/R}\right)\times$$

$$\times\sum_{k,r=1}^{+\infty}\frac{1}{g^2_{kr}}\frac{k\pi}{a}\left(\frac{r\pi}{b}\right)^2 \sin^2\left(\frac{k\pi}{2}\right)\cos^2\left(\frac{r\pi}{2}\right)\left\{\frac{\cos\frac{r\pi}{2}}{\mathrm{sh}\left(\frac{k\pi b}{2a}\right)} - \frac{\cos\frac{k\pi}{2}}{\mathrm{ch}\left(\frac{k\pi a}{2b}\right)}\right\}. \quad (31)$$

To get the possibility to sum up (31) over k,r we must take into account the transverse dimension of a bunch:

$$j_z = I_o\frac{e^{-(x-x_o)^2/2\Delta_x^2}}{\sqrt{2\pi}\,\Delta_x}\,\delta(y)\delta(z-z_o), \qquad x_o, \Delta_x \ll b.$$

Then the r-th item in (31) must be multiplied on $e^{-\left(\frac{r\pi\Delta x}{R}\right)^2/2}$, and after summing over r the sum over k,r in (31) transforms into the next expression (in the limit $\Delta_x \to 0$):

$$\sum_{k,r=1}^{+\infty} = \frac{1}{\sqrt{ab}}F_x\left(\frac{b}{a}\right) = \frac{1}{b}\sum_{k=0}^{\infty}\left\{\sin\left(\frac{k\pi}{2}\right)\frac{\frac{k\pi b}{2a}}{\mathrm{sh}\left(\frac{k\pi b}{2a}\right)}\right\}^2 +$$

$$+\frac{1}{a}\sum_{k=0}^{\infty}\left\{\cos\left(\frac{k\pi}{2}\right)\frac{\frac{k\pi a}{2b}}{\mathrm{sh}\left(\frac{k\pi a}{2b}\right)}\right\}^2.$$

A function $F_x(t)$ is shown on a fig. 2b.

The equation of the transverse oscillations has a form (18). In the first approach we can neglect synchrotron oscillations (z_o = const) and a shift of a transverse oscillations frequency ($\psi_x = \omega_x t$).

A beam with a full current I_o consisting of n_o similar macroparticles symmetrically placed along the orbit has n_o symmetrical modes of transverse oscillations with the phase shifts between neighbour macroparticles $2\pi(k-1)/n_o$, $k=1,\dots,n_o$.

After summing up the currents of all bunches and averaging over the time the r.h.s. of eq. (18) we get the growth rate of the k-th mode of the transverse oscillations:

$$\sigma_x^k = -\frac{\omega_o I_o}{\pi V_s \nu_x} \sum_{p=-\infty}^{+\infty} \frac{Re(Z_x(i(pn_o+k)\omega_o+i\Omega_x))}{(pn_o+k)+\nu_x}, \quad k=1,\dots,n_o, \quad (32)$$

$$Z_x(s)=2\pi R\cdot\frac{R^2}{(ab)^{3/2}}\cdot F_x\left(\frac{b}{a}\right)\cdot\xi(s)=\frac{\pi R^2\delta_o}{(ab)^{3/2}} F_x\left(\frac{b}{a}\right) Z_o\cdot\sqrt{2s/\omega_o},$$

$$V_s=m_s c^2/e .$$

For one bunch with a current I_o

$$\sigma_{xo}=-\frac{\omega_o I_o}{V_s \nu_x}\frac{\sqrt{2}R^2\delta_o}{(ab)^{3/2}} F_x\left(\frac{b}{a}\right)Z_o\left\{\frac{1}{\sqrt{\nu'_x}}+\sum_{m=1}^{+\infty}\frac{1}{\sqrt{m+\nu'_x}}-\frac{1}{\sqrt{m-\nu'_x}}\right\}, \quad (33)$$

where ν'_x is the fractional portion of ν_x, $0 \le \nu'_x < 1$.

It is easy to show that the single bunch growth rate is positive when $\nu'_x > 0.5$ and negative when $\nu'_x < 0.5$.

In the multibunch regime the growth rate of the k-th mode can be written as

$$\sigma_x^k = -\frac{\omega_o I_o}{V_s \nu_x}\frac{\sqrt{2}R^2\delta_o}{(ab)^{3/2}} F_x\left(\frac{b}{a}\right)Z_o\frac{S_k}{\sqrt{n_o}}, \quad (34)$$

where

$$S_k=\frac{1}{\sqrt{f_k}}+\sum_{p=1}^{+\infty}\frac{1}{\sqrt{p+f_k}}-\frac{1}{\sqrt{p-f_k}}, \quad f_k=\frac{k+\nu'_x}{n_o}. \quad (35)$$

S_k has the same form as a series in the curly brackets in (33), but f_k changes for different modes from $\frac{\nu'_x}{n_o}$ up to $1-\frac{1-\nu'_x}{n_o}$. Thus the multibunch beam has always the oscillations modes both with positive (at $f_k>0.5$) and with negative (at $f_k<0.5$) growth rates (fig. 3b).

Note that when increasing f from 0 up to 1 S(f) decreases from $+\infty$ down to $-\infty$; S(0.5)=0; S(f)=-S(1-f). The growth rate is maximum at the maximum value of f:

$$S\left(1-\frac{1-\nu'_x}{n_o}\right)=S\left(\frac{1-\nu'_x}{n_o}\right)\approx\sqrt{\frac{n_o}{1-\nu'_x}}.$$

Thus the maximum growth rate in the multibunch regime is proportional to $\sqrt{\frac{1}{1-\nu'_x}}$, that is the fractional portion of ν_x must be possibly less. When $\nu'_x \ll 1$ the maximum growth rate can be estimated as

$$\sigma_x = \frac{\omega_o I_o}{V_s \nu_x}\frac{\sqrt{2}R^2\delta_o}{(ab)^{3/2}} F_x\left(\frac{b}{a}\right)Z_o\approx 180 \text{ s}^{-1}. \quad (36)$$

Radiation damping of the transverse oscillations ξ_x is 4 times less than that of longitudinal oscillations, $\xi_x\approx 100$ s^{-1}. So the transverse resistive instability is not compensated by this damping, and there arises a problem of supporting the transverse stability of the beam by some other means.

5. The Transverse Instability of a Beam with a Gap

Let's consider the symmetrical beam consisting of n_o equal bunches with a current of each bunch I_1 and a full current $I_o=I_1\cdot n_o$. If we take away one or some bunches from this beam the maximum growth rate of its oscillations will decrease. If when decreasing the number of bunches n the maximum growth rate decreases faster than the full current of a beam $I=I_1\cdot n$, then this gap in a beam can be some way to suppress the instability. In other words the condition given above can be formulated as decreasing the maximum growth rate when decreasing the number of bunches in a beam by maintaining the full current of a beam.

To determine the maximum growth rate of the nonsymmetrical beam we will now develop the method used above, because of absence of the symmetry we have to solve the both equations (18) for all bunches simultaneously. When calculating a Lorentz force (31) we must sum up it over all bunches taking into account:

a) a longitudinal position of each bunch, $z_k = 2\pi/n_o \cdot (k-1)$, $k=1,\dots,n \le n_o$;

b) unknown phases of the bunches, $\hat{\psi}_k = \Omega_x t + \psi_k$, $k=1,\dots,n \le n_o$;

c) different amplitudes of the bunches $y_k = \sqrt{J_k}$.

It is convenient to turn to the complex amplitudes $t_k = y_k e^{i\psi_k}$. Then the system of equations (18) can be easy transformed to the matrix equation:

$$\hat{A}\,\hat{S}\,\hat{N}\,\vec{t} = \lambda\,\vec{t}\,, \qquad (37)$$

where

$$A = \omega_o \frac{2R^3}{(ab)^{3/2}} \frac{F_x(b/a)}{\nu_x V_s} \sqrt{Z_o \rho/R}\;;$$

$$N_{ik} = I_i \delta_{ik},\; i=1,\dots,n\;; \qquad \lambda = \sigma + i\cdot\Delta\Omega\;;$$

$$S_{jk} = \sum_{m=-\infty}^{+\infty} \exp(i\, m\, 2\pi/n_o \cdot (j-k)) \frac{\sqrt{-i(m-\nu_x)}}{(m-\nu_x)}\,. \qquad (38)$$

Note that here $\sqrt{-i(m-\nu_x)} = \frac{1\mp i}{\sqrt{2}}\sqrt{|m-\nu_x|}$ for $(m-\nu_x) \gtrless 0$, that corresponds to the positive sign of the impedance real part for all m.

If j=k a series (38) can be summed up as shown above for the symmetrical beam. If j≠k this series can be expressed via integrals given in Appendix 3. So, all the elements of the matrix S can be easy calculated.

A computation of the eigenvalues of the system (37) leads to the next results.

1. Fig.4a represents a maximum growth rate of the transverse oscillations σ in the dependence of the relative length of the beam n/n_o for different deviations of the integer resonances ν'_x. Here we mean the current of one bunch to be constant, that is the full current to be proportional to the length of the beam. Fig. 4b represents the dependence of the symmetrical beam maximum growth rate σ_o on the deviation from the integer resonance In these calculations $n_o=140$.

2. Fig. 5a represents the results given on the fig. 4a normalized on the equal full current for each case, it is the dependence of $\sigma_n = \frac{\sigma/\sigma_o}{n/n_o}$ on the n/n_o. In the logarithmic scale (fig. 5b; the abscisse is $-\ln(1-\sigma_n)$, the ordinate is $-\ln(1-n/n_o)$) these dependences are practically straight lines parallel one another with a slope approximately equal 2. The slopes of the lines and the shifts between them are given in the fig. 6a,b. Thus the dependence of the maximum growth rate for the small gaps can be approximated as

$$\sigma = \sigma_o \cdot n/n_o (1 - A\cdot(1-n/n_o)^{\alpha}), \text{ where } \alpha \approx 2,\; A \approx 2\, e^{-1.2\nu'_x}.$$

(For $n_o \ne 140$ the coefficient A will be changed, see fig. 7, 8.)

3. Fig. 7a,b represents a dependence of the normalized maximal growth rate on the relative beam length for $\nu'_x=0.1$ and different n_o in linear(a) and logarithmic(b) scale analogously to the fig.5a,b. The slopes of the logarithmic lines and the shifts between them are given in the fig. 8a,b analogously to the fig. 6a,b.

Conclusion

We have shown that the longitudinal resistive instability of the multibunch beam is negligibly small in comparison with a single bunch regime. But the growth rates of the transverse resistive instability appear to exceed a synchrotronous damping. A possibility to decrease the transverse instability by using a nonsymmetrical beam with a gap has been analyzed, and it was shown that it is not a way to suppress sufficiently the resistive instability.

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Appendix 1

The Eigenfunctions of the Waveguide

$$E_k^{(E)}(x,y,s)=-\vec{\nabla}\phi_k+\frac{g_k^2}{\gamma_k}\phi_k \mathbf{1}_o\,,\quad E_k^{(H)}(x,y,s)=\mathbf{1}_o\times\vec{\nabla}\psi_k\,,$$

$$H_k^{(E)}(x,y,s)=-\frac{s\varepsilon_o}{\gamma_k}\mathbf{1}_o\times\vec{\nabla}\phi_k\,,\quad H_k^{(H)}(x,y,s)=-\frac{\gamma'_k}{s\mu_o}\vec{\nabla}\psi_k+\frac{g_k^2}{s\mu_o}\psi_k\mathbf{1}_o\,,$$

where $$\Delta\phi_k+g_k^2\phi_k=0\,,\qquad \phi_k\big|_C=0\,;$$

$$\Delta\psi_k+g_k'^2\psi_k=0\,,\qquad \frac{\partial\psi_k}{\partial n}\Big|_C=0\,;$$

At the normalization $g_k^2\int\limits_S\phi_k^2\,dS=1\,,\quad g_k'^2\int\limits_S\psi_k^2\,dS=1\,,$

we have $D_k^E=s\varepsilon_o/\gamma_k\,,\ D_k^H=\gamma'_k/s\mu_o\,.$

The propagation number is determined as

$$\gamma_k^2=g_k^2+s_k^2/c^2\,.$$

Appendix 2

The Coefficients in the Functions Φ_m, Ψ_m.

If we denote $$A_m=-\frac{\xi/2}{2\pi R}\,\frac{im/R}{((m/R)^2+s^2\mu\,\varepsilon)}\,,$$

$$J_{km}=\frac{I_{km}}{\gamma_k+im/R}-\frac{I_{-km}}{\gamma_k-im/R}\,,$$

$$J'_{km}=\frac{2im/R}{\gamma_k'^2+(m/R)^2}\,I'_{km}\,,$$

then the coefficients in (13), (14) can be written as

$$\begin{Bmatrix}M'_k\\ N'_k\end{Bmatrix}=\frac{A_m}{h_{mkx}\begin{Bmatrix}ch\\ sh\end{Bmatrix}\left(\frac{h_{mkx}b}{2}\right)}\sum_r\frac{\frac{k\pi}{a}\left(\frac{r\pi}{b}J_{krm}+\frac{k\pi}{a}J'_{krm}\right)}{g_{kr}\sqrt{ab}}(1\mp(-1)^r),$$

$$\begin{Bmatrix}M''_r\\ N''_r\end{Bmatrix}=\frac{A_m}{h_{mry}\begin{Bmatrix}ch\\ sh\end{Bmatrix}\left(\frac{h_{mry}a}{2}\right)}\sum_k\frac{\frac{r\pi}{b}\left(-\frac{k\pi}{a}J_{krm}+\frac{r\pi}{b}J'_{krm}\right)}{g_{kr}\sqrt{ab}}(1\mp(-1)^k),$$

$$\begin{Bmatrix} C'_k \\ D'_k \end{Bmatrix} = \frac{A_m}{\begin{Bmatrix} sh \\ ch \end{Bmatrix}\left(\frac{h_{mkx}\, b}{2}\right)} \sum_r \frac{\left(\frac{r\pi}{b} J_{krm} + \frac{k\pi}{a} J'_{krm}\right)}{g_{kr}\sqrt{ab}} \left(1 \pm (-1)^r\right) ,$$

$$\begin{Bmatrix} C''_r \\ D''_r \end{Bmatrix} = \frac{A_m}{\begin{Bmatrix} sh \\ ch \end{Bmatrix}\left(\frac{h_{mry}\, a}{2}\right)} \sum_k \frac{\left(\frac{k\pi}{a} J_{krm} - \frac{r\pi}{b} J'_{krm}\right)}{g_{kr}\sqrt{ab}} \left(1 \pm (-1)^k\right).$$

Appendix 3

A Transformation of the Series (38) to the Integrals

The series (38) can be easy turn to the next series which can be transformed to the integrals ([5], formula 5.4.3(1))·

$$\sum_{k=0}^{\infty} \frac{1}{(k+q)^s} \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-qt}}{1-2e^{-t}\cos(x)+e^{-2t}} \times$$

$$\times \left(\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \pm e^{-t} \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix} \right) dt ,$$

where $s=0.5$, $q=\nu'_x$.

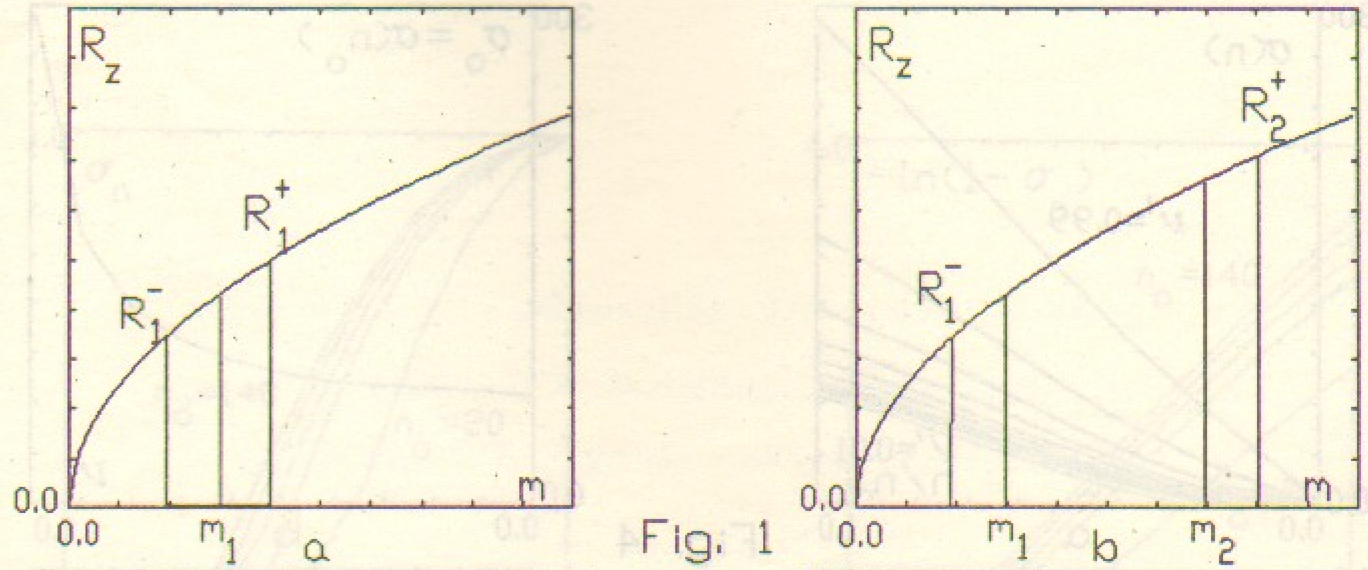


Fig. 1

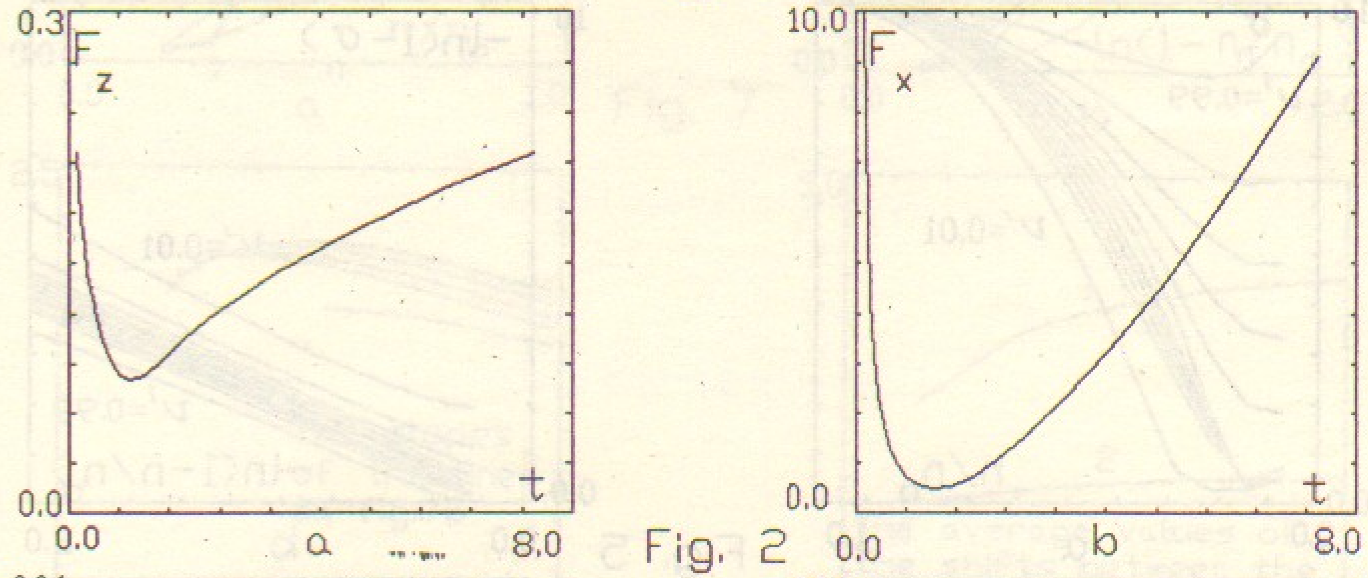


Fig. 2

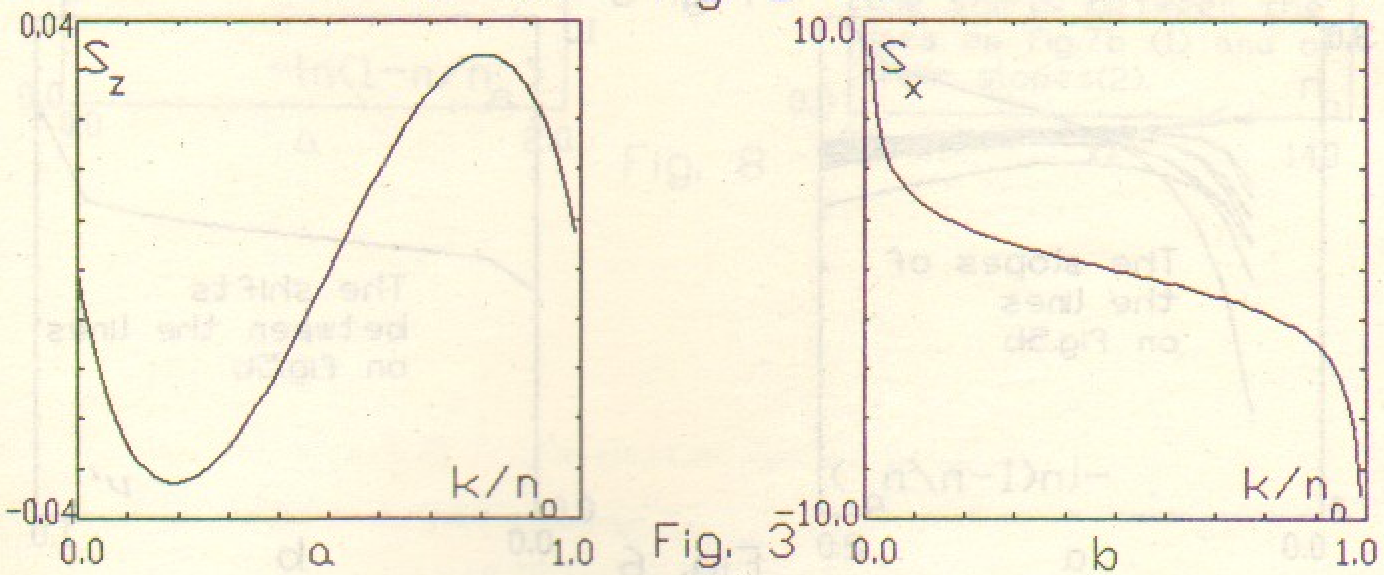


Fig. 3

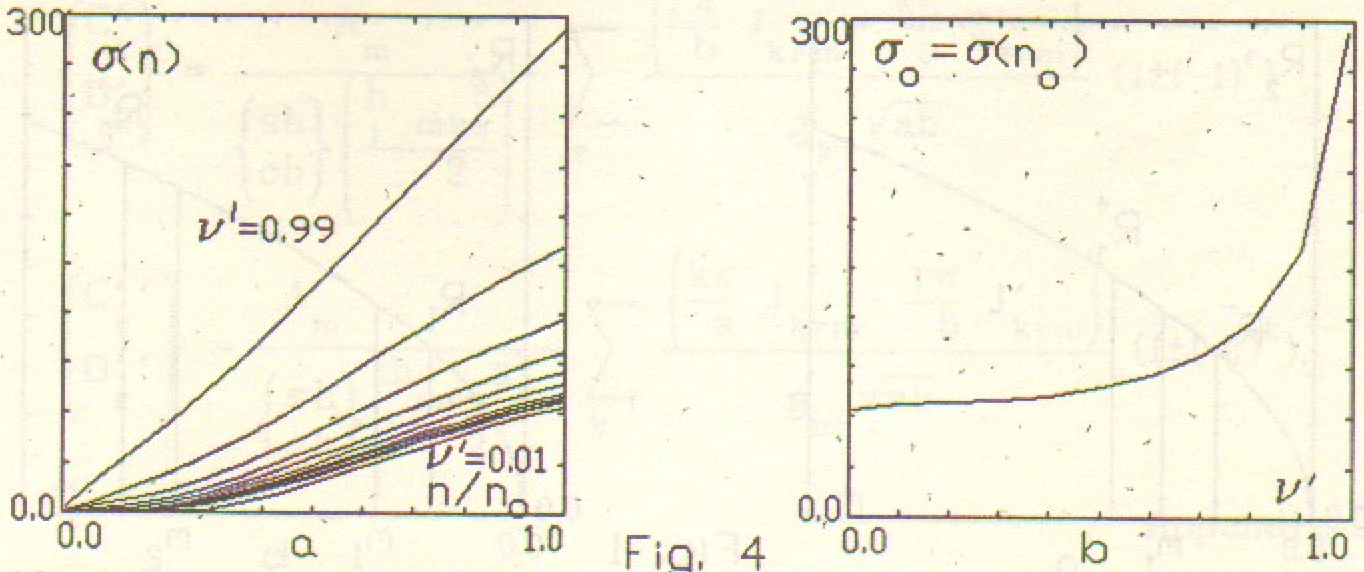


Fig. 4

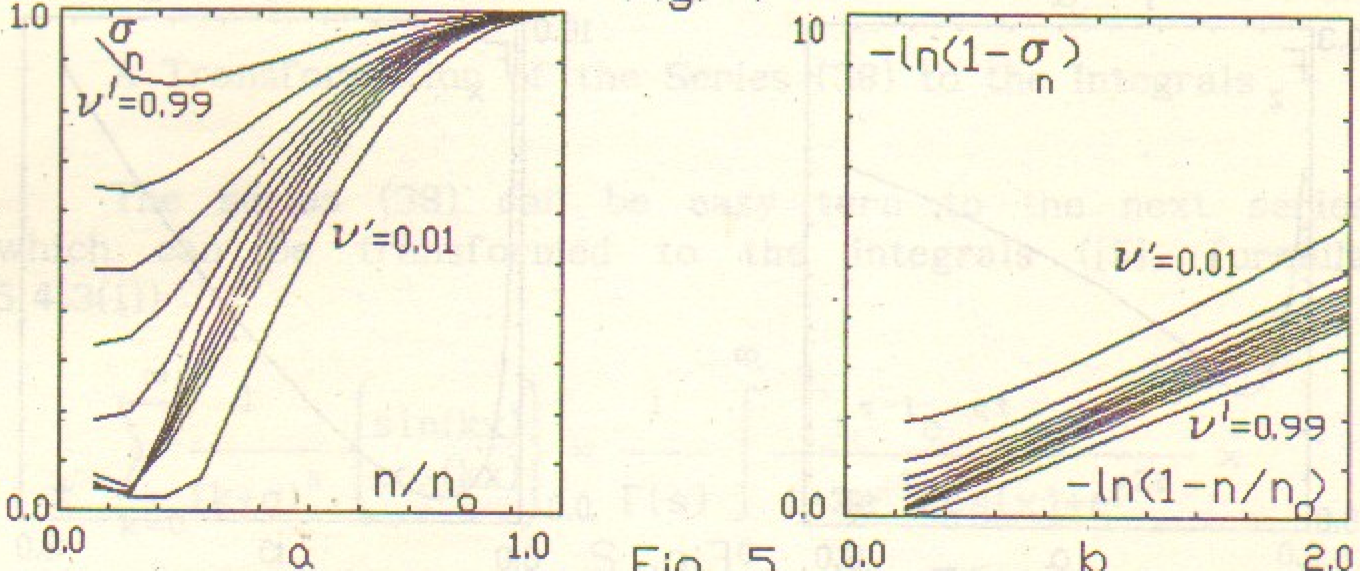
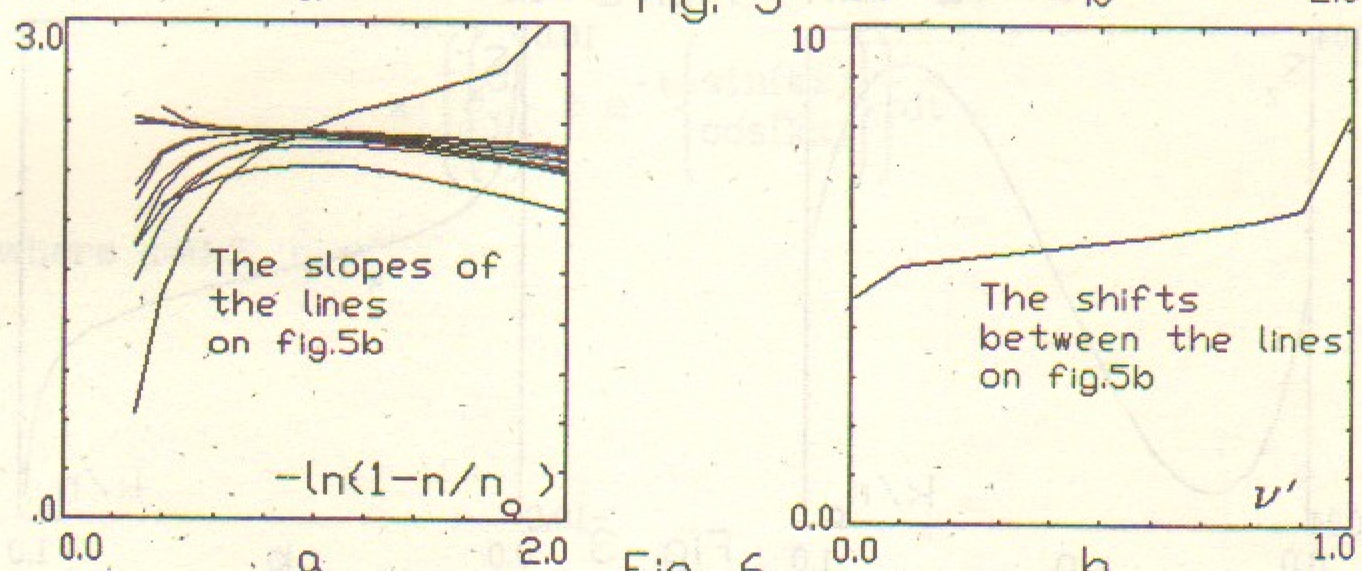


Fig. 5



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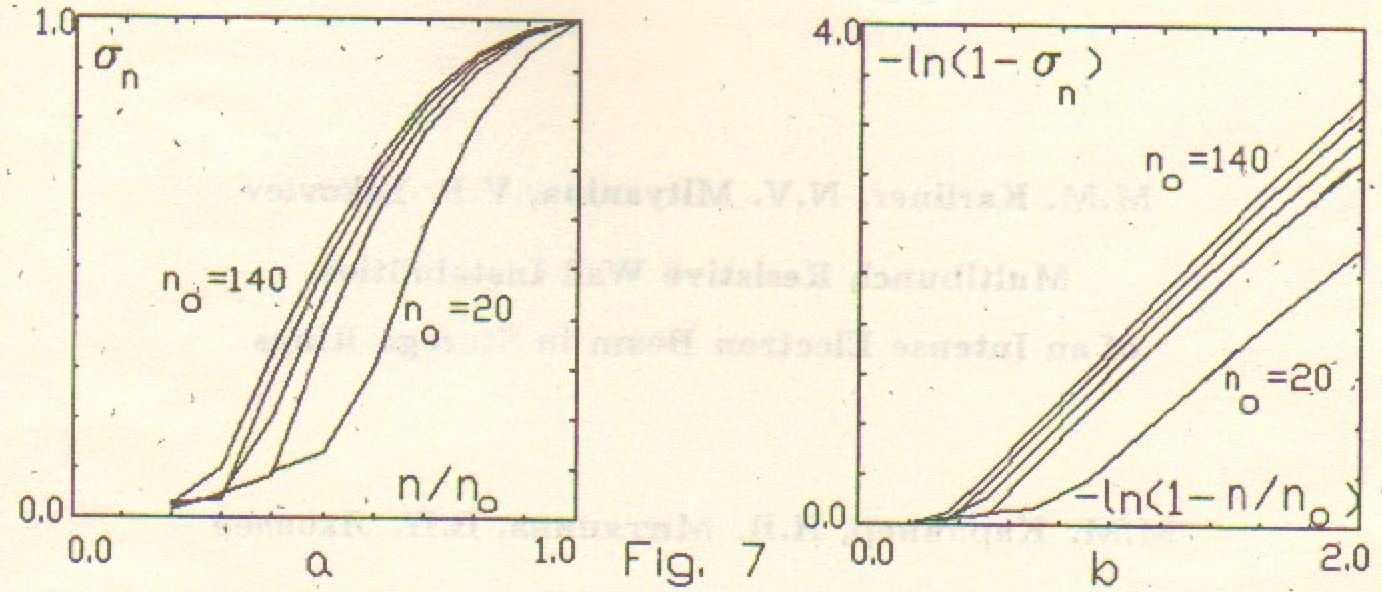

Fig. 7

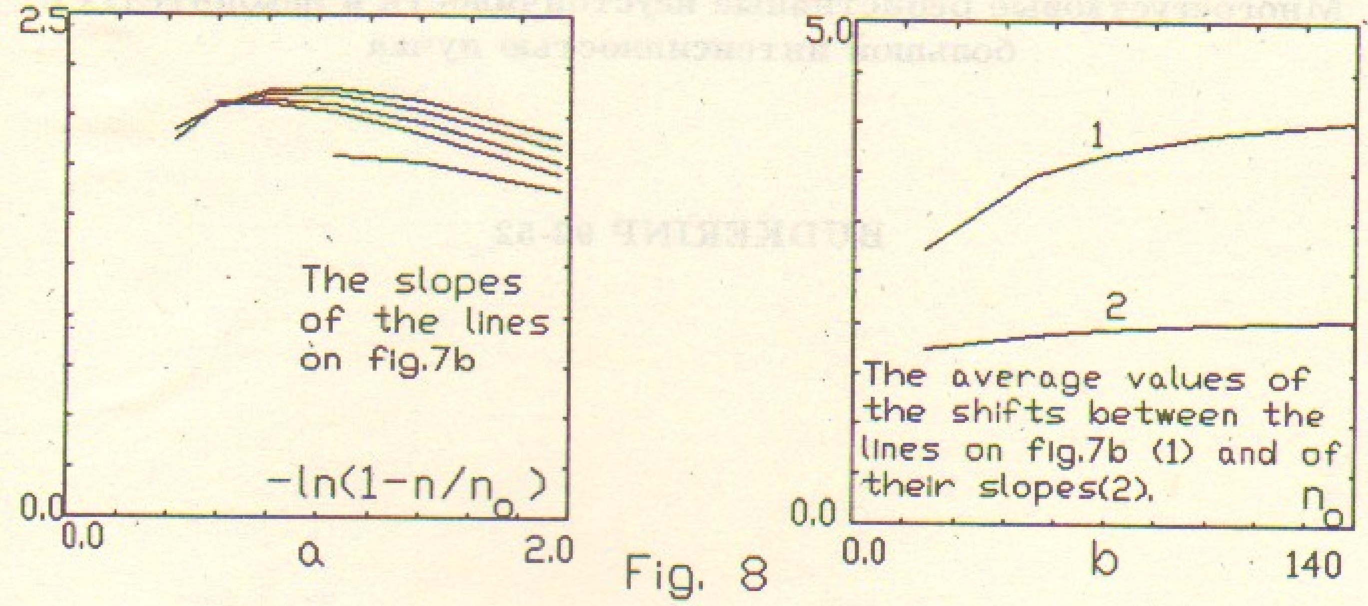

Fig. 8