ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

A.G. Grozin

METHODS OF CALCULATION OF HIGHER POWER CORRECTIONS IN QCD

PREPRINT 92-11

НОВОСИБИРСК

Methods of Calculation

of Higher Power Corrections in QCD

A. G. Grozin

Budker Institute of Nuclear Physics
630090, Novosibirsk-90, Russia

ABSTRACT

We review the algorithms of calculation of higher power corrections in QCD sum rules based on the fixed-point gauge.

1 Introduction

The Operator Product Expansion (OPE) is widely used for investigation of correlators in the quantum field theory. In particular, it is the basis of the QCD sum rules method. Let $\Pi(p)$ be a correlator of two local currents in the momentum space. Then OPE reads

$$\Pi(p) = \sum_i c_i(p^2) \langle O_i \rangle , \tag{1}$$

where $\langle O_i \rangle$ are vacuum averages of local operators of various dimensions d (called also vacuum condensates), and $c_i(p^2)$ are coefficient functions. They are perturbative series in α_s. In order to make sum rules more precise and to establish their applicability regions it is necessary to calculate both higher perturbative corrections (higher terms in α_s expansions of coefficient functions) and higher non-perturbative (power) corrections. We shall not discuss physical applications of QCD sum rules here.

Methods of calculation of higher loop diagrams (perturbative corrections) form an established field of research, and many articles and reviews are devoted to them. We shall not discuss this subject; instead we shall concentrate on the methods of calculation of higher-dimensional terms in the OPE (1) in the leading order in α_s.

There already exists an excellent review on technical problems of QCD sum rules calculations [1]. The present article can be considered as an addendum to it. It is based on the works devoted to calculation of higher power corrections: the heavy quark case was considered in [2], and the light quark one—in [3, 4, 5] and [6, 7]. We shall formulate the algorithms of calculation of higher power corrections in a form suitable for a Computer Algebra implementation. Full technical details (partly omitted from the original papers) will be included for the reader's convenience. In all cases when a question is discussed in [1] we shall refer to this review and the literature cited in it.[1]

Unfortunately, there is no complete package for such calculations in conventional Computer Algebra systems (REDUCE, Mathematica,...). Such a package would be very useful. I have written several Modula-2 programs which implement some of the discussed algorithms and produce a REDUCE readable output. They are useful as a set of tools though they can't be a substitute for a complete package.

The plan of the review is following. In the Section 2 we shall briefly discuss the fixed-point (Fock-Schwinger) gauge on which the modern methods of calculation of power corrections are based; for more details and proofs see [1]. In the Section 3 we shall discuss the systematic classification of vacuum condensates. There exist many relations between condensates, therefore we should choose a basis of independent condensates and formulate an algorithm of reducing an arbitrary condensate to this basis. Heavy quark condensates and correlators are discussed in the Section 4, and light quark ones—in the Section 5.

2 Fixed-point gauge

The QCD vacuum has a non-trivial structure. In order to calculate a correlator in QCD, we should first calculate it in a background (vacuum) gluon and quark field. Then we should average the expression for a correlator via these fields over the vacuum. After that we obtain the expression for a correlator via vacuum condensates (1).

Correlators of colourless currents are gauge invariant. Therefore we can use any gauge for the background gluon field. It is convenient to use the fixed-point (Fock-Schwinger) gauge

$$x_\mu A^a_\mu(x) = 0. \qquad (2)$$

In this gauge the Taylor expansions for $A^a_\mu(x)$ and $q(x)$ can be written in a gauge-covariant form [1]

$$A^a_\mu(x) = \frac{1}{2\cdot 0!} x_\nu G^a_{\nu\mu}(0) + \frac{1}{3\cdot 1!} x_\alpha x_\nu D_\alpha G^a_{\nu\mu}(0)$$

[1] I should like to apologize to the authors of these articles for not citing them directly.

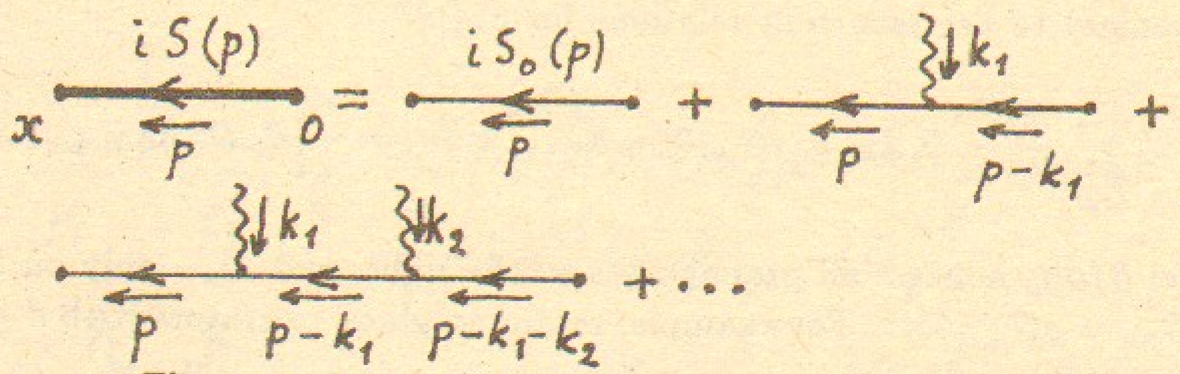

Figure 1: Quark propagator in background gluon field

$$+ \frac{1}{4\cdot 2!} x_\beta x_\alpha x_\nu D_\beta D_\alpha G^a_{\nu\mu}(0) + \cdots, \qquad (3)$$

$$q(x) = q(0) + x_\alpha D_\alpha q(0) + \frac{1}{2!} x_\beta x_\alpha D_\beta D_\alpha q(0) + \cdots$$

(and hence $\overline{q}(x) = \overline{q}(0) + \overline{q}(0)\overleftarrow{D}_\alpha x_\alpha + \cdots$). Here $D_\mu q = (\partial_\mu - iA_\mu)q$ is a covariant derivative in the fundamental representation, $A_\mu = gA^a_\mu t^a$; $D_\mu G^a_{\rho\sigma} = (\partial_\mu \delta^{ab} - iA^{ab}_\mu) G^b_{\rho\sigma}$ is a covariant derivative in the adjoint representation, $A^{ab}_\mu = gf^{acb}A^c_\mu$. Hence only gauge covariant quantities are used at all stages of calculations.

In this gauge the theory is not translational invariant. A translation should be accompanied by a gauge transformation to another fixed point gauge. Correlators become gauge invariant and hence translational invariant after vacuum averaging. We can choose the gauge fixed point (origin in (2)) at any vertex of the correlator to simplify calculations. In the momentum space, background gluon lines have incoming momenta k_i and contribute $A^a_\mu(k_i)\frac{d^4k_i}{(2\pi)^4}$ where $A^a_\mu(k_i)$ is the Fourier transform of (3) and is the series in derivatives of $\delta(k_i)$. The vertex chosen as the gauge fixed point provides a common sink for all vacuum momenta k_i.

The quark propagator (Fig. 1) can be written as a sum over the number of background gluon lines:

$$S(p) = \sum_{k=0}^{\infty} S_k(p), \quad S_0(p) = \frac{\hat{p}+m}{p^2-m^2}, \quad S_k(p) = -S_0(p)\hat{A}S_{k-1}(p),$$

$$A_\mu = -i\sum_{l=0}^{\infty} \frac{(-i)^l}{(l+2)l!} D_{\alpha_1}\dots D_{\alpha_l} G^a_{\nu\mu} t^a \partial_{\alpha_1}\dots\partial_{\alpha_l}\partial_\nu. \qquad (4)$$

It is convenient to use recurrent relations for $S_k(p)$:

$$S_k = S_0 \sum_{l=0}^{\infty} \frac{A_{kl}}{l+2}, \quad A_{k0} = iG_{\mu\nu}\partial_\mu\gamma_\nu S_{k-1}, \quad A_{kl} = -\frac{i}{l}\partial_\alpha D_\alpha A_{k,l-1}. \quad (5)$$

Here $\partial_\mu = \partial/\partial p_\mu$ acts on all propagators to the right, and D_μ—only on the nearest $G_{\mu\nu} = gG^a_{\mu\nu}(0)t^a$. For example, retaining gluon operators with $d \le 4$ (Fig. 1) we have

$$\begin{aligned} S &= S_0 + \frac{i}{2}G_{\mu\nu}S_0\partial_\mu\gamma_\nu S_0 + \frac{1}{3}D_\alpha G_{\mu\nu}S_0\partial_\alpha\partial_\mu\gamma_\nu S_0 \\ &- \frac{i}{8}D_\beta D_\alpha G_{\mu\nu}S_0\partial_\beta\partial_\alpha\partial_\mu\gamma_\nu S_0 - \frac{1}{4}G_{\rho\sigma}G_{\mu\nu}S_0\partial_\rho\gamma_\sigma S_0\partial_\mu\gamma_\nu S_0. \end{aligned} \quad (6)$$

I have written a Modula-2 program for simplification of expressions constructed from S_0, ∂_μ, and γ_μ. It applies ∂_μ to all S_0 to the right using $\partial_\mu S_0 = -S_0\gamma_\mu S_0$, collects similar terms, and produces a REDUCE readable output. It can be used, for example, to expand these expressions in basis γ matrices.

These formulae can be used for both massive and massless quarks. For a light quark we can expand S_0 in m. For a massless quark the equation (6) gives

$$\begin{aligned} S(p) &= \frac{1}{p^2}\hat{p} - \frac{1}{p^4}p_\mu\tilde{G}_{\mu\nu}\gamma_\nu\gamma_5 \\ &+ \frac{1}{p^6}\left[-\frac{2}{3}\left(p^2\hat{J} - p_\mu J_\mu\hat{p} + p_\lambda D_\lambda p_\mu G_{\mu\nu}\gamma_\nu\right) - 2ip_\lambda D_\lambda p_\mu\tilde{G}_{\mu\nu}\gamma_\nu\gamma_5\right] \\ &+ \frac{1}{p^8}\Bigg[2\left(p^2\gamma_\mu G_{\mu\lambda}G_{\lambda\nu}p_\nu - p_\mu G_{\mu\lambda}G_{\lambda\nu}p_\nu\hat{p}\right) - 2ip_\lambda D_\lambda\left(p^2\hat{J} - p_\mu J_\mu\hat{p}\right) \\ &+ \left(4(p_\lambda D_\lambda)^2 - p^2D^2\right)p_\mu\tilde{G}_{\mu\nu}\gamma_\nu\gamma_5 - i\left[p_\mu G_{\mu\lambda}, \tilde{G}_{\lambda\nu}\gamma_\nu\gamma_5\right]\Bigg]. \end{aligned} \quad (7)$$

Here $J_\mu = gJ^a_\mu t^a$, $J^a_\mu = D_\nu G^a_{\mu\nu} = g\sum_{q'}\bar{q}'\gamma_\mu t^a q'$. Note that the last term in (7) is missing in [1]. In deriving this formula we used the relations $D^2G_{\mu\nu} = -D_\mu J_\nu + D_\nu J_\mu - 2i[G_{\mu\lambda}, G_{\lambda\nu}]$, $D_\nu D_\lambda G_{\mu\nu} = D_\lambda J_\mu - i[G_{\nu\lambda}, G_{\mu\nu}]$, $G_{\mu\lambda}\tilde{G}_{\lambda\nu} + \tilde{G}_{\nu\lambda}G_{\lambda\mu} = -\frac{1}{2}\delta_{\mu\nu}\tilde{G}_{\rho\sigma}G_{\rho\sigma}$, $G_{\mu\nu}\tilde{G}_{\mu\nu} = \tilde{G}_{\mu\nu}G_{\mu\nu}$. In practical calculations it is often more convenient to substitute symbolic expressions for propagators like (6) to the diagrams and to simplify the complete answers than to use (7).

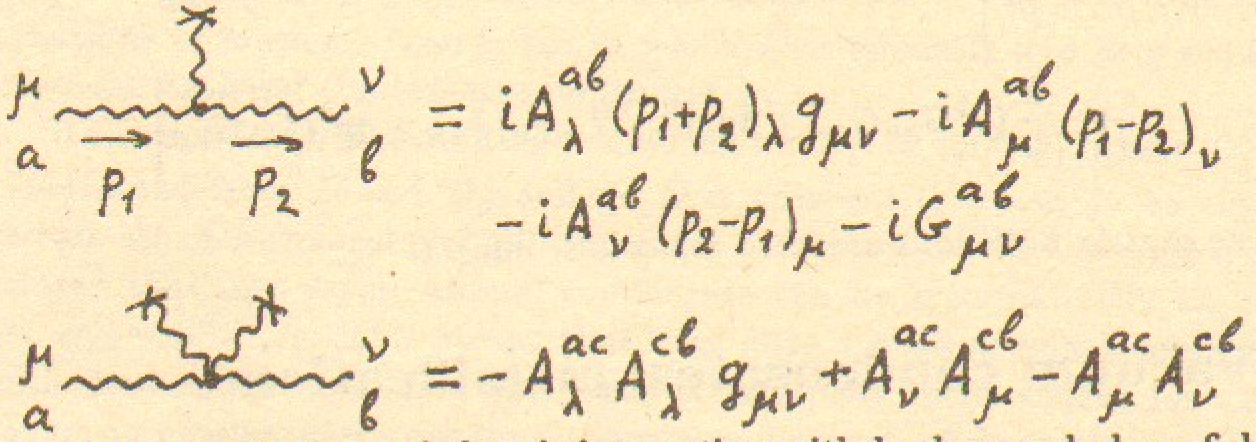

Figure 2: Vertices of gluon's interaction with background gluon field

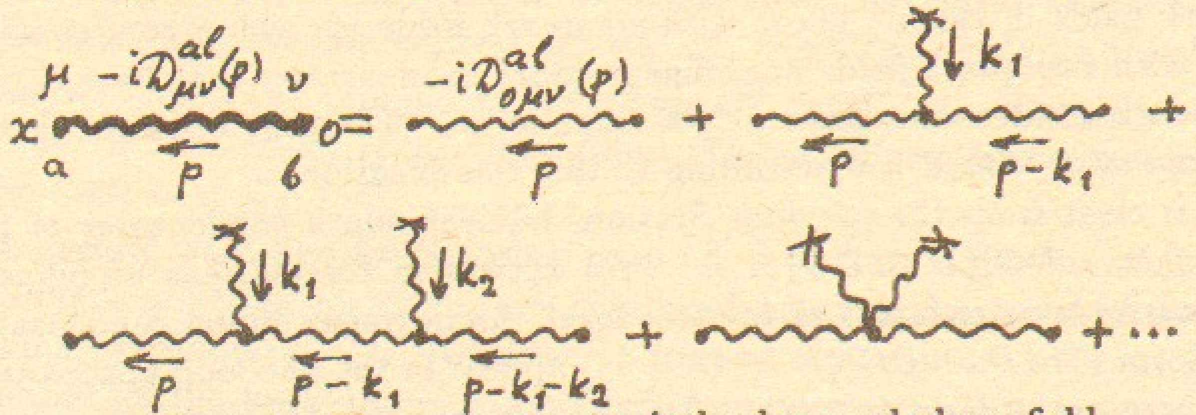

Figure 3: Gluon propagator in background gluon field

In order to consider the gluon propagation in the background gluon field we should substitute $A^a_\mu + a^a_\mu$ into the QCD lagrangian, where A^a_μ is the background field and a^a_μ is the quantum gluon field. The lagrangian becomes

$$L = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} - \frac{1}{2}(D_\mu a^a_\nu)(D_\mu a^a_\nu) + \frac{1}{2}(D_\mu a^a_\mu)(D_\nu a^a_\mu) + \frac{1}{2}a^a_\mu G^{ab}_{\mu\nu}a^b_\nu + \cdots \quad (8)$$

Here $G^a_{\mu\nu}$ and D_μ include only the background field A^a_μ, and $G^{ab}_{\mu\nu} = gf^{acb}G^c_{\mu\nu}$. We choose the fixed-point gauge for the background field A^a_μ. This lagrangian is still gauge invariant with respect to a^a_μ. We should add a gauge fixing term and a corresponding ghost term. It is convenient to use a generalization of the Feynman gauge fixing term $-\frac{1}{2}(D_\mu a^a_\mu)^2$. Then the free gluon propagator is $D^0_{\mu\nu}(p) = \delta_{\mu\nu}/p^2$, and the vertices of the gluon's interaction with the background field (Fig. 2) are $iA^{ab}_\lambda(p_1+p_2)_\lambda\delta_{\mu\nu} - iA^{ab}_\mu(p_1-p_2)_\nu - iA^{ab}_\nu(p_2-p_1)_\mu - iG^{ab}_{\mu\nu}$ and $-A^{ac}_\lambda A^{cb}_\lambda\delta_{\mu\nu} - A^{ac}_\mu A^{cb}_\nu + A^{ac}_\mu A^{cb}_\nu$.

Retaining terms with $d \le 4$, we have the gluon propagator (Fig. 3)

$$D_{\alpha\beta} = \frac{1}{p^2}\delta_{\alpha\beta} + \frac{1}{p^4}2G_{\alpha\beta} + \frac{1}{p^6}\left(\frac{2}{3}ip_\mu J_\mu\delta_{\alpha\beta} + 4ip_\lambda D_\lambda G_{\alpha\beta}\right)$$

$$+ \frac{1}{p^8}\Bigg[-2p_\lambda D_\lambda p_\mu J_\mu \delta_{\alpha\beta} - 2\left(4(p_\lambda D_\lambda)^2 - p^2 D^2\right) G_{\alpha\beta} \qquad (9)$$

$$+ \frac{1}{2}\left(p^2 G_{\mu\nu} G_{\mu\nu} + 4 p_\mu G_{\mu\lambda} G_{\lambda\nu} p_\nu\right)\delta_{\alpha\beta} + 4p^2 G_{\alpha\lambda} G_{\lambda\beta}\Bigg].$$

Here the matrix notations are used for colour indices.

3 Vacuum condensates' classification

Vacuum condensates can be divided into classes according to the number of quark fields in them. Those without quark fields are gluon condensates. Those with two quark fields are bilinear quark condensates; they have $d \geq 3$. Four-quark condensates have $d \geq 6$, and so on. The unit operator is the gluon operator with $d = 0$ according to this classification.

As is clear from the previous Section, bilinear quark condensates of the form $\langle \overline{q}(D\ldots DG)(D\ldots DG)\ldots\gamma\ldots\gamma q\rangle$ appear in calculations of bilinear quark currents' correlators at the tree level. Analogously, gluon condensates of the form $\langle \mathrm{Tr}\,(D\ldots DG)(D\ldots DG)\ldots\rangle$ appear in the one-loop approximation (there are at least two $(D\ldots DG)$ groups because otherwise the colour trace vanishes).

In the Section 3.1 we shall discuss the systematic classification of bilinear quark condensates following [7]. We shall present all formulae necessary for the practical use of this method. In the Section 3.2 we shall apply similar methods to the gluon condensates. In the Section 3.3 we shall discuss methods of vacuum averaging of expressions for correlators via quark and gluon fields.

3.1 Bilinear quark condensates

Using the formulae $G_{\mu\nu} = gG^a_{\mu\nu}t^a = i[D_\mu, D_\nu]$, $(D_\mu A) = (D^{ab}_\mu A^b)t^a = [D_\mu, A]$ (where A^a is in the adjoint representation of the colour group, and $A = A^a t^a$), we can easily reduce any bilinear quark condensate of dimension $d = m + 3$ to a combination of terms of the form $\langle \overline{q} D_{\mu_1}\ldots D_{\mu_m}\Gamma_{\mu_1\ldots\mu_m} q\rangle$, where $\Gamma_{\mu_1\ldots\mu_m}$ is constructed from γ_μ and $\delta_{\mu\nu}$. Choosing the terms with the largest number of γ matrices, we permute them in such a way that their indices are in the same order as in $D_{\mu_1}\ldots D_{\mu_m}$. Arising additional terms have fewer γ matrices. We repeat this procedure going to terms with fewer γ matrices until we reach terms with 1 γ matrix or without them at all. As a result, a bilinear quark condensate reduces to a linear combination of terms

of the form $\langle \overline{q} O_i q\rangle$, where O_i are constructed from $\hat{D}$ and D_μ. Due to the equations of motion, those terms in which $\hat{D}$ is adjacent to q or $\overline{q}$ reduce to lower-dimensional condensates multiplied by m.

Having written down all the condensates $B^d_i = \langle \overline{q} O^m_i q\rangle$ where O^m_i are constructed from $\hat{D}$ and D_μ and no $\hat{D}$ is adjacent to q and $\overline{q}$, we obtain a certain set of d-dimensional bilinear quark condensates. We have just demonstrated that any d-dimensional condensate can be systematically expressed via B^d_i, $mB^{d-1}_i, \ldots$ This means that they form a basis of quark condensates with dimensions $\leq d$. But this basis is extremely inconvenient for use because its condensates contain a maximum number of derivatives acting on q. So we should better choose a basis of the most convenient d-dimensional condensates Q^d_j. We can express Q^d_j via B^d_i, $mB^{d-1}_i, \ldots$ Solving this linear system we obtain the expressions for B^d_i via Q^d_j, $mQ^{d-1}_j, \ldots$ Having these expressions we can easily reduce any d-dimensional bilinear quark condensate to the basis ones Q^d_j, $mQ^{d-1}_j, \ldots$

A general guideline for choosing good Q^d_j is to have a minimum number of derivatives acting on the quark field, and hence a maximum dimension of gluon operators in Q^d_j. It allows to expand propagators to a minimum dimension in the background gluon field. Among such operators one should first of all choose those containing J_μ because they are really four-quark ones and are more easily calculable. They may be suppressed in some vacuum models (of the instanton type) in which vacuum gluon fields may be strong but $D_\nu G_{\mu\nu} \approx 0$.

For dimensions $d \leq 6$ we have

$$B^3 = \langle \overline{q}q\rangle\,, \quad B^5 = -\langle \overline{q}D^2 q\rangle\,, \quad B^6 = -i\left\langle \overline{q} D_\mu \hat{D} D_\mu q\right\rangle. \qquad (10)$$

We choose the basis condensates

$$Q^3 = \langle \overline{q}q\rangle\,, \quad Q^5 = i\langle \overline{q} G_{\mu\nu}\sigma_{\mu\nu} q\rangle\,, \quad Q^6 = \left\langle \overline{q}\hat{J} q\right\rangle, \qquad (11)$$

where $\sigma_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]}$, and square brackets mean antisymmetrization. These condensates are expressed via B^d_i as

$$\begin{aligned} Q^3 &= B^3, \\ Q^5 &= -2B^5 + 2m^2 B^3, \\ Q^6 &= -2B^6 + 2mB^5. \end{aligned} \qquad (12)$$

Solving this linear system we obtain

$$B^3 = Q^3,$$

$$
\begin{aligned}
B^5 &= -\frac{1}{2}Q^5 + m^2 Q^3, \\
B^6 &= -\frac{1}{2}Q^6 - \frac{m}{2}Q^5 + m^3 Q^3.
\end{aligned} \tag{13}
$$

For $d = 7$ we have

$$
\begin{aligned}
&B_1^7 = \langle \bar q D^2 D^2 q\rangle, \quad B_2^7 = \langle \bar q D_\mu D^2 D_\mu q\rangle, \\
&B_3^7 = \langle \bar q D_\mu D_\nu D_\mu D_\nu q\rangle, \quad B_4^7 = \left\langle \bar q D_\mu \hat D^2 D_\mu q\right\rangle.
\end{aligned} \tag{14}
$$

We choose basis condensates

$$
\begin{aligned}
&Q_1^7 = \langle \bar q G_{\mu\nu} G_{\mu\nu} q\rangle, \quad Q_2^7 = i\left\langle \bar q G_{\mu\nu}\tilde G_{\mu\nu}\gamma_5 q\right\rangle, \\
&Q_3^7 = \langle \bar q G_{\mu\lambda} G_{\lambda\nu}\sigma_{\mu\nu} q\rangle, \quad Q_4^7 = i\langle \bar q D_\mu J_\nu \sigma_{\mu\nu} q\rangle.
\end{aligned} \tag{15}
$$

Here the condensate containing $\tilde G_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}$ and $\gamma_5 = \frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta$ is understood as a short notation for the expression from which both ε tensors are eliminated using $\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta} = -4!\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]}$, and this expression is valid at any space dimension D. These condensates are expressed via B_i^d as

$$
\begin{aligned}
Q_1^7 &= 2B_2^7 - 2B_3^7, \\
Q_2^7 &= -2B_1^7 - 2B_2^7 + 2B_3^7 + 2B_4^7 - 4mB^6 + 6m^2 B^5 - 2m^4 B^3, \\
Q_3^7 &= 2B_2^7 - 2B_3^7 - B_4^7 + 2mB^6 - m^2 B^5, \\
Q_4^7 &= -2B_1^7 - 2B_2^7 + 4B_3^7 - 4mB^6 + 4m^2 B^5.
\end{aligned} \tag{16}
$$

Solving this linear system we obtain

$$
\begin{aligned}
B_1^7 &= \frac{1}{2}Q_1^7 - \frac{1}{2}Q_2^7 - Q_3^7 - m^2 Q^5 + m^4 Q^3, \\
B_2^7 &= \frac{3}{2}Q_1^7 - \frac{1}{2}Q_2^7 - Q_3^7 + \frac{1}{2}Q_4^7 - mQ^6 - m^2 Q^5 + m^4 Q^3, \\
B_3^7 &= Q_1^7 - \frac{1}{2}Q_2^7 - Q_3^7 + \frac{1}{2}Q_4^7 - mQ^6 - m^2 Q^5 + m^4 Q^3, \\
B_4^7 &= Q_1^7 - Q_3^7 - mQ^6 - \frac{m^2}{2}Q^5 + m^4 Q^3.
\end{aligned} \tag{17}
$$

Similarly, for $d = 8$ we have

$$
B_1^8 = i\left\langle \bar q(D_\mu \hat D D_\mu D^2 + D^2 D_\mu \hat D D_\mu) q\right\rangle,
$$

$$
\begin{aligned}
&B_2^8 = i\left\langle \bar q(D_\mu \hat D D^2 D_\mu + D_\mu D^2 \hat D D_\mu) q\right\rangle, \\
&B_3^8 = i\left\langle \bar q(D_\mu \hat D D_\nu D_\mu D_\nu + D_\nu D_\mu D_\nu \hat D D_\mu) q\right\rangle, \\
&B_4^8 = i\left\langle \bar q D^2 \hat D D^2 q\right\rangle, \quad B_5^8 = i\left\langle \bar q D_\mu D_\nu \hat D D_\nu D_\mu q\right\rangle, \\
&B_6^8 = i\left\langle \bar q D_\mu D_\nu \hat D D_\mu D_\nu q\right\rangle, \quad B_7^8 = i\left\langle \bar q D_\mu \hat D^3 D_\mu q\right\rangle.
\end{aligned} \tag{18}
$$

Operators similar to those in $B_{1\ldots3}^8$ but with a minus sign between two terms are C-odd, and their vacuum averages vanish.

An interesting new phenomenon similar to the axial anomaly arises for bilinear quark condensates of dimensions $d \geq 8$. At a space dimension $D \neq 4$ we can construct the condensate

$$
A = i\langle \bar q D_\alpha D_\beta D_\gamma D_\delta D_\varepsilon \gamma_{[\alpha}\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_{\varepsilon]} q\rangle. \tag{19}
$$

All coefficients in the expansion of this condensate in gluon condensates (see the Section 4.1) contain traces vanishing in the 4-dimensional space. Momentum integrals in coefficients at $d > 8$ gluon condensates converge, and at $d = 8$ they contain divergencies $1/\varepsilon$. As a result, the quark condensate A is equal to a combination of $d = 8$ gluon condensates.

Choosing the anomalous condensate A as one of the basis ones, we can choose 6 proper quark $d = 6$ basis condensates as

$$
\begin{aligned}
&Q_1^8 = i\langle \bar q[[G_{\mu\lambda}, G_{\lambda\nu}]_+, D_\mu]_+ \gamma_\nu q\rangle, \\
&Q_2^8 = -\left\langle \bar q[[G_{\mu\lambda}, \tilde G_{\lambda\nu}], D_\mu]_+ \gamma_\nu\gamma_5 q\right\rangle, \\
&Q_3^8 = i\left\langle \bar q[\hat D G_{\mu\nu}, G_{\mu\nu}] q\right\rangle, \quad Q_4^8 = \left\langle \bar q D^2 \hat J q\right\rangle, \\
&Q_5^8 = i\langle \bar q[G_{\mu\nu}, J_\mu]\gamma_\nu q\rangle, \quad Q_6^8 = \left\langle \bar q[\tilde G_{\mu\nu}, J_\mu]_+ \gamma_\nu\gamma_5 q\right\rangle.
\end{aligned} \tag{20}
$$

Here again the expressions containing $\tilde G_{\mu\nu}$ and $\gamma_\mu\gamma_5 = \frac{i}{3!}\varepsilon_{\mu\alpha\beta\gamma}\gamma_\alpha\gamma_\beta\gamma_\gamma$ are understood as a short notation for the expressions in which both ε tensors are eliminated.

All operators in (20) (as well as in (15), (11)) are purely C-even. C-conjugation permutes all D_μ and γ_μ in the opposite order, and changes their signs (because D_μ is transformed to $\overleftarrow{D}_\mu$). Therefore every commutator gives a factor -1 to the C-parity, and every anticommutator gives $+1$; we should not use ordinary products if we want to obtain operators with a definite C-parity. According to these rules, $G_{\mu\nu}$ and all its derivatives $D_\lambda G_{\mu\nu}, \ldots$

(including J_μ) are C-odd; γ_μ and $\sigma_{\mu\nu}$ are C-odd, while 1, γ_5, and $\gamma_\mu\gamma_5$ are C-even.

The condensates (19), (20) are expressed via (18) as

$$\begin{aligned}
A &= -B_1^8 - B_2^8 + B_3^8 + B_4^8 + B_5^8 - B_6^8 + B_7^8 \\
&+ 2mB_1^7 + 2mB_2^7 - 2mB_3^7 - 2mB_4^7 + 3m^2B^6 - 4m^3B^5 + m^5B^3, \\
Q_1^8 &= B_1^8 - B_2^8 + 2B_5^8 - 2B_6^8 - 2mB_1^7 + 2mB_3^7, \\
Q_2^8 &= 2B_1^8 - 2B_2^8 - 2B_4^8 + 2B_5^8 - 2B_6^8 + 2B_7^8 \\
&- 2mB_1^7 + 2mB_2^7 + 2mB_3^7 - 2mB_4^7 - 2m^2B^6 + 2m^3B^5, \\
Q_3^8 &= -4B_5^8 + 4B_6^8 + 4mB_2^7 - 4mB_3^7, \\
Q_4^8 &= 2B_1^8 + 2B_2^8 - 2B_4^8 - 4B_5^8 - 2mB_1^7, \\
Q_5^8 &= B_1^8 + B_2^8 - 2B_3^8 - 2mB_1^7 - 2mB_2^7 + 4mB_3^7, \\
Q_6^8 &= 3B_1^8 + B_2^8 - 2B_3^8 - 2B_4^8 \\
&- 4mB_1^7 - 2mB_2^7 + 4mB_3^7 - 4m^2B^6 + 4m^3B^5.
\end{aligned} \tag{21}$$

Solving this linear system we obtain

$$\begin{aligned}
B_1^8 &= A + \frac{1}{2}Q_1^8 - \frac{1}{2}Q_2^8 + \frac{1}{4}Q_3^8 - \frac{1}{2}Q_5^8 + Q_6^8 \\
&+ mQ_1^7 - \frac{m}{2}Q_2^7 - 2mQ_3^7 - m^2Q^6 - 2m^3Q^5 + 2m^5Q^3, \\
B_2^8 &= A - \frac{1}{2}Q_1^8 - \frac{1}{2}Q_2^8 - \frac{1}{4}Q_3^8 - \frac{1}{2}Q_5^8 + Q_6^8 \\
&+ 3mQ_1^7 - \frac{m}{2}Q_2^7 - 2mQ_3^7 + mQ_4^7 - 3m^2Q^6 - 2m^3Q^5 + 2m^5Q^3, \\
B_3^8 &= A - \frac{1}{2}Q_2^8 - Q_5^8 + Q_6^8 \\
&+ 2mQ_1^7 - \frac{m}{2}Q_2^7 - 2mQ_3^7 + mQ_4^7 - 3m^2Q^6 - 2m^3Q^5 + 2m^5Q^3, \\
B_4^8 &= A + \frac{1}{2}Q_1^8 - \frac{1}{2}Q_2^8 + \frac{1}{4}Q_3^8 + \frac{1}{2}Q_6^8 \\
&+ \frac{m}{2}Q_1^7 - mQ_3^7 - m^3Q^5 + m^5Q^3, \qquad (22) \\
B_5^8 &= \frac{1}{2}A - \frac{1}{4}Q_1^8 - \frac{1}{4}Q_2^8 - \frac{1}{8}Q_3^8 - \frac{1}{4}Q_4^8 - \frac{1}{2}Q_5^8 + \frac{3}{4}Q_6^8 \\
&+ \frac{3}{2}mQ_1^7 - \frac{m}{4}Q_2^7 - mQ_3^7 + \frac{m}{2}Q_4^7 - 2m^2Q^6 - m^3Q^5 + m^5Q^3, \\
B_6^8 &= \frac{1}{2}A - \frac{1}{4}Q_1^8 - \frac{1}{4}Q_2^8 + \frac{1}{8}Q_3^8 - \frac{1}{4}Q_4^8 - \frac{1}{2}Q_5^8 + \frac{3}{4}Q_6^8
\end{aligned}$$

$$\begin{aligned}
&+ mQ_1^7 - \frac{m}{4}Q_2^7 - mQ_3^7 + \frac{m}{2}Q_4^7 - 2m^2Q^6 - m^3Q^5 + m^5Q^3, \\
B_7^8 &= A - \frac{1}{2}Q_1^8 + \frac{1}{2}Q_6^8 \\
&+ mQ_1^7 + \frac{m}{2}Q_2^7 - mQ_3^7 - \frac{3}{2}m^2Q^6 - \frac{m^3}{2}Q^5 + m^5Q^3.
\end{aligned}$$

I have implemented the discussed algorithm as a Modula-2 program. It accepts an arbitrary bilinear quark condensate and expresses it via B_i^d producing a REDUCE readable output. It can be used to express the condensate via Q_j^d using the formulae (13), (17), (22).

The problem of four-quark classification is much more difficult. Even at $d = 6$ there exist infinitely many condensates $\langle\bar{q}\gamma_{[\mu_1}\cdots\gamma_{\mu_n]}q\rangle\langle\bar{q}\gamma_{\mu_1}\cdots\gamma_{\mu_n]}q\rangle$, $\langle\bar{q}t^a\gamma_{[\mu_1}\cdots\gamma_{\mu_n]}q\rangle\langle\bar{q}t^a\gamma_{\mu_1}\cdots\gamma_{\mu_n]}q\rangle$. Those with $d > 4$ are anomalous and can be eliminated [8].

3.2 Gluon condensates

In this Section we apply similar methods to the classification of gluon condensates. For each dimension $d = 2n$ we introduce a linear space. We shall call it the extended space. It is generated by the basis of formal sequences E_i^d constructed from D_μ in which each index is contained twice. Sequences obtained from each other by renaming indices, cyclic permutations (trace cyclicity), and reversing (C-parity) are considered equivalent. Physical gluon condensates can be systematically expressed via this basis, and form a subspace of the extended space. We can choose a basis in this subspace composed from the most convenient condensates G_j^d, and extend it to a basis in the whole space using a subset of E_i^d. After that we can expand any condensate in E_i^d and then reexpress it via G_j^d and the selected E_i^d. In fact E_i^d should not appear if we have indeed selected a complete basis in the physical subspace.

General guidelines for choosing good basis condensates are similar to the quark case. First of all, we should include all independent condensates containing J_μ. The number of derivatives should be kept minimum.

There is only one gluon condensate with $d = 4$, namely $G^4 = \langle \mathrm{Tr}\, G_{\mu\nu}G_{\mu\nu}\rangle$. The extended space is two-dimensional: $E_1^4 = D^2D^2$, $E_2^4 = D_\mu D_\nu D_\mu D_\nu$; $G^4 = 2E_1^4 - 2E_2^4$. But all this is of no use here.

At $d = 6$ the extended space is 5-dimensional:

$$\begin{aligned}
&E_1^6 = D^2D^2D^2, \quad E_2^6 = D^2D_\mu D_\nu D_\mu D_\nu, \quad E_3^6 = D^2D_\mu D^2 D_\mu, \\
&E_4^6 = D_\lambda D_\mu D_\lambda D_\nu D_\mu D_\nu, \quad E_5^6 = D_\lambda D_\mu D_\nu D_\lambda D_\mu D_\nu.
\end{aligned} \tag{23}$$

There are 2 linearly independent physical condensates:

$$G_1^6 = i\langle \mathrm{Tr}\, G_{\lambda\mu} G_{\mu\nu} G_{\nu\lambda} \rangle, \quad G_2^6 = \langle \mathrm{Tr}\, J_\mu J_\mu \rangle. \tag{24}$$

They are expressed via E_i^6 as

$$\begin{aligned} G_1^6 &= E_1^6 - 3E_2^6 + 3E_4^6 - E_5^6, \\ G_2^6 &= -2E_1^6 + 8E_2^6 - 2E_3^6 - 4E_4^6. \end{aligned} \tag{25}$$

We can exclude $E_{4\ldots5}^6$:

$$\begin{aligned} E_4^6 &= -\frac{1}{4}G_2^6 - \frac{1}{2}E_1^6 + 2E_2^6 - \frac{1}{2}E_3^6, \\ E_5^6 &= -G_1^6 - \frac{3}{4}G_2^6 - \frac{1}{2}E_1^6 + 3E_2^6 - \frac{3}{2}E_3^6. \end{aligned} \tag{26}$$

After that we can expand any $d = 6$ condensate in (23) and express it via (24) and $E_{1\ldots3}^6$; the last terms should not appear. We have verified it for all $d = 6$ condensates.

At $d = 8$ the extended space is 17-dimensional:

$$\begin{aligned} &E_1^8 = D^2 D^2 D^2 D^2, \quad E_2^8 = D^2 D^2 D_\mu D_\nu D_\mu D_\nu, \\ &E_3^8 = D^2 D^2 D_\mu D^2 D_\mu, \quad E_4^8 = D^2 D_\lambda D_\mu D_\lambda D_\nu D_\mu D_\nu, \\ &E_5^8 = D^2 D_\lambda D_\mu D_\lambda D^2 D_\mu, \quad E_6^8 = D^2 D_\lambda D_\mu D_\nu D_\lambda D_\mu D_\nu, \\ &E_7^8 = D^2 D_\lambda D_\mu D_\nu D_\lambda D_\nu D_\mu, \quad E_8^8 = D^2 D_\lambda D_\mu D_\nu D_\mu D_\nu D_\lambda, \\ &E_9^8 = D^2 D_\mu D_\nu D^2 D_\mu D_\nu, \quad E_{10}^8 = D^2 D_\mu D_\nu D^2 D_\nu D_\mu, \\ &E_{11}^8 = D_\mu D_\nu D_\mu D_\nu D_\rho D_\sigma D_\rho D_\sigma, \quad E_{12}^8 = D_\mu D_\nu D_\mu D_\rho D_\nu D_\sigma D_\rho D_\sigma, \\ &E_{13}^8 = D_\mu D_\nu D_\mu D_\rho D_\sigma D_\nu D_\rho D_\sigma, \quad E_{14}^8 = D_\mu D_\nu D_\mu D_\rho D_\sigma D_\nu D_\sigma D_\rho, \\ &E_{15}^8 = D_\mu D_\nu D_\rho D_\mu D_\sigma D_\nu D_\rho D_\sigma, \quad E_{16}^8 = D_\mu D_\nu D_\rho D_\mu D_\sigma D_\rho D_\nu D_\sigma, \\ &E_{17}^8 = D_\mu D_\nu D_\rho D_\sigma D_\mu D_\nu D_\rho D_\sigma. \end{aligned} \tag{27}$$

A set of independent $d = 8$ gluon condensates was found in [2]. We choose one condensate in a different way [7]:

$$\begin{aligned} &G_1^8 = \langle \mathrm{Tr}\, G_{\mu\nu} G_{\mu\nu} G_{\alpha\beta} G_{\alpha\beta} \rangle, \quad G_2^8 = \langle \mathrm{Tr}\, G_{\mu\nu} G_{\alpha\beta} G_{\mu\nu} G_{\alpha\beta} \rangle, \\ &G_3^8 = \langle \mathrm{Tr}\, G_{\mu\alpha} G_{\alpha\nu} G_{\nu\beta} G_{\beta\mu} \rangle, \quad G_4^8 = \langle \mathrm{Tr}\, G_{\mu\alpha} G_{\alpha\nu} G_{\mu\beta} G_{\beta\nu} \rangle, \\ &G_5^8 = i\langle \mathrm{Tr}\, J_\mu G_{\mu\nu} J_\nu \rangle, \quad G_6^8 = i\langle \mathrm{Tr}\, J_\lambda [D_\lambda G_{\mu\nu}, G_{\mu\nu}] \rangle, \\ &G_7^8 = \langle \mathrm{Tr}\, J_\mu D^2 J_\mu \rangle. \end{aligned} \tag{28}$$

All condensates are written in an explicitly C-even form (even number of commutators). They are expressed via E_i^8 as

$$\begin{aligned} G_1^8 &= -8E_8^8 + 4E_{10}^8 + 4E_{11}^8, \\ G_2^8 &= -8E_{15}^8 + 4E_{16}^8 + 4E_{17}^8, \\ G_3^8 &= E_1^8 - 4E_2^8 + 4E_4^8 + 2E_{11}^8 - 4E_{12}^8 + E_{16}^8, \\ G_4^8 &= 2E_6^8 - 4E_7^8 + E_9^8 + 2E_{12}^8 - 4E_{13}^8 + 2E_{14}^8 + E_{15}^8, \\ G_5^8 &= E_1^8 - 5E_2^8 + 2E_3^8 + 4E_4^8 - 2E_5^8 + 4E_7^8 - 4E_8^8 - E_9^8 + E_{10}^8 \\ &+ 4E_{11}^8 - 4E_{12}^8, \\ G_6^8 &= -4E_2^8 + 4E_3^8 + 8E_6^8 - 8E_7^8 - 12E_8^8 + 4E_{10}^8 + 8E_{11}^8 - 8E_{13}^8 + 8E_{14}^8, \\ G_7^8 &= -2E_1^8 + 8E_2^8 - 6E_3^8 - 8E_4^8 + 12E_5^8 - 16E_7^8 + 4E_9^8 + 8E_{14}^8. \end{aligned} \tag{29}$$

We can exclude $E_{11\ldots17}^8$:

$$\begin{aligned} E_{11}^8 &= \frac{1}{4}G_1^8 + 2E_8^8 - E_{10}^8, \\ E_{12}^8 &= \frac{1}{4}G_1^8 - \frac{1}{4}G_5^8 \\ &+ \frac{1}{4}E_1^8 - \frac{5}{4}E_2^8 + \frac{1}{2}E_3^8 + E_4^8 - \frac{1}{2}E_5^8 + E_7^8 + E_8^8 - \frac{1}{4}E_9^8 - \frac{3}{4}E_{10}^8, \\ E_{13}^8 &= \frac{1}{4}G_1^8 - \frac{1}{8}G_6^8 + \frac{1}{8}G_7^8 + \frac{1}{4}E_1^8 - \frac{3}{2}E_2^8 \\ &+ \frac{5}{4}E_3^8 + E_4^8 - \frac{3}{2}E_5^8 + E_6^8 + E_7^8 + \frac{1}{2}E_8^8 - \frac{1}{2}E_9^8 - \frac{1}{2}E_{10}^8, \\ E_{14}^8 &= \frac{1}{8}G_7^8 + \frac{1}{4}E_1^8 - E_2^8 + \frac{3}{4}E_3^8 + E_4^8 - \frac{3}{2}E_5^8 + 2E_7^8 - \frac{1}{2}E_9^8, \\ E_{15}^8 &= \frac{1}{2}G_1^8 + G_4^8 + \frac{1}{2}G_5^8 - \frac{1}{2}G_6^8 + \frac{1}{4}G_7^8 \\ &- \frac{3}{2}E_2^8 + \frac{5}{2}E_3^8 - 2E_5^8 + 2E_6^8 + 2E_7^8 - \frac{3}{2}E_9^8 - \frac{1}{2}E_{10}^8, \\ E_{16}^8 &= \frac{1}{2}G_1^8 + G_3^8 - G_5^8 - E_2^8 + 2E_3^8 - 2E_5^8 + 4E_7^8 - E_9^8 - E_{10}^8, \\ E_{17}^8 &= \frac{1}{2}G_1^8 + \frac{1}{4}G_2^8 - G_3^8 + 2G_4^8 + 2G_5^8 - G_6^8 + \frac{1}{2}G_7^8 \\ &- 2E_2^8 + 3E_3^8 - 2E_5^8 + 4E_6^8 - 2E_9^8. \end{aligned} \tag{30}$$

After that we can expand any $d = 8$ condensate in (27) and express it via (28) and $E_{1\ldots10}^8$; the last terms should not appear.

I have written a Modula-2 program that expands any gluon condensate in E_i^d and produces a REDUCE readable output that can be used for expressing this condensate via G_j^d and selected E_i^d. Using this program, I have verified that all $d = 8$ gluon condensates indeed can be expanded in the basis (28).

3.3 Vacuum averaging

After obtaining an expression for a correlator via background fields we should average it over the vacuum. The first possible way is to write down the most general expressions for vacuum averages with free indices and to find unknown coefficients by solving linear systems.

For the quark condensates with $d \leq 6$ we have

$$\begin{aligned}
\langle \bar{q}_\sigma q_\rho \rangle &= \frac{Q^3}{2^2} 1_{\rho\sigma}, \\
\langle \bar{q}_\sigma D_\alpha q_\rho \rangle &= -\frac{imQ^3}{2^4} (\gamma_\alpha)_{\rho\sigma}, \\
\langle \bar{q}_\sigma D_\alpha D_\beta q_\rho \rangle &= \frac{1}{2^5} \left[Q^5 \left(\delta_{\alpha\beta} + \frac{i}{3}\sigma_{\alpha\beta} \right) - 2m^2 Q^3 \delta_{\alpha\beta} \right]_{\rho\sigma}, \qquad (31) \\
\langle \bar{q}_\sigma D_\alpha D_\beta D_\gamma q_\rho \rangle &= \frac{1}{2^6 3^3} \Big[Q^6 (\delta_{\alpha\beta}\gamma_\gamma + \delta_{\beta\gamma}\gamma_\alpha - 5\delta_{\alpha\gamma}\gamma_\beta - 3i\varepsilon_{\alpha\beta\gamma\delta}\gamma_\delta\gamma_5) \\
&- 3mQ^5 (\delta_{\alpha\beta}\gamma_\gamma + \delta_{\beta\gamma}\gamma_\alpha + \delta_{\alpha\gamma}\gamma_\beta - i\varepsilon_{\alpha\beta\gamma\delta}\gamma_\delta\gamma_5) \\
&+ 6m^3 Q^3 (\delta_{\alpha\beta}\gamma_\gamma + \delta_{\beta\gamma}\gamma_\alpha + \delta_{\alpha\gamma}\gamma_\beta) \Big]_{\rho\sigma}.
\end{aligned}$$

One can easily obtain similar formulae with $G_{\alpha\beta}$ and its derivatives by antisymmetrization over corresponding indices.

For the gluon condensates with $d \leq 6$ we have

$$\begin{aligned}
\langle g^2 G^a_{\mu\nu} G^b_{\rho\sigma} \rangle &= \frac{G^4}{NC_F D(D-1)} \delta^{ab} (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}), \\
\langle g^2 G^a_{\mu\nu} D_\alpha D_\beta G^b_{\rho\sigma} \rangle &= \frac{\delta^{\alpha\beta}}{NC_F D(D-1)} \Big[-\frac{G_1^6}{D-2} A_{\mu\nu\alpha\beta\rho\sigma} \\
&+ \frac{4G_1^6 - 2G_2^6}{D} \delta_{\alpha\beta} (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) \Big], \qquad (32) \\
\langle g^3 G^a_{\mu\nu} G^b_{\alpha\beta} G^c_{\rho\sigma} \rangle &= -\frac{2iG_1^6}{N^2 C_F D(D-1)(D-2)} f^{abc} A_{\mu\nu\alpha\beta\rho\sigma},
\end{aligned}$$

$$\begin{aligned}
A_{\mu\nu\alpha\beta\rho\sigma} &= \delta_{\nu\alpha}\delta_{\beta\rho}\delta_{\sigma\mu} - \delta_{\mu\alpha}\delta_{\beta\rho}\delta_{\sigma\nu} - \delta_{\nu\beta}\delta_{\alpha\rho}\delta_{\sigma\mu} - \delta_{\nu\alpha}\delta_{\beta\sigma}\delta_{\rho\mu} \\
&+ \delta_{\mu\beta}\delta_{\alpha\rho}\delta_{\sigma\nu} + \delta_{\mu\alpha}\delta_{\beta\sigma}\delta_{\rho\nu} + \delta_{\nu\beta}\delta_{\alpha\sigma}\delta_{\rho\mu} - \delta_{\mu\beta}\delta_{\alpha\sigma}\delta_{\rho\nu}.
\end{aligned}$$

We can substitute these relations to expressions for correlators and obtain scalar expressions. But these formulae become very complicated in higher dimensions.

An alternative and simpler way is to average an expression for a correlator over p directions. This is done using the recurrent relation (n is even)

$$\begin{aligned}
\overline{p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n}} &= \frac{p^2}{D+n-2} (\delta_{\mu_1\mu_2} \overline{p_{\mu_3} \cdots p_{\mu_n}} + \delta_{\mu_1\mu_3} \overline{p_{\mu_2} \cdots p_{\mu_n}} + \cdots \\
&+ \delta_{\mu_1\mu_n} \overline{p_{\mu_2} \cdots p_{\mu_{n-1}}}) \qquad (33)
\end{aligned}$$

or the explicit formula

$$\overline{p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n}} = \frac{(p^2)^{n/2}}{D(D+2)\ldots(D+n-2)} (\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4} \cdots \delta_{\mu_{n-1}\mu_n} + \cdots), \qquad (34)$$

where the sum is taken over all $n!!$ methods of paring n indices. Here D is the space dimension; the formulae can be used at any dimension including non-integer one (the dimensional regularization). After such an averaging we obtain scalar vacuum condensates for which the methods of the previous Sections can be directly used.

In calculations of three-current correlators we encounter the averages $\overline{p_{\mu_1} \cdots p_{\mu_m} q_{\nu_1} \cdots q_{\nu_n}}$ over orientations of the pair p, q with a fixed relative orientation. These averages can be calculated using the decomposition $q = (qp)p/p^2 + q_\perp$, $q_\perp^2 = (p^2q^2 - (pq)^2)/p^2$. First we average over orientations of $q_\perp$ orthogonal to p in $(D-1)$-dimensional space using (34) with $\delta_{\mu\nu} \to \delta_{\perp\mu\nu} = \delta_{\mu\nu} - p_\mu p_\nu/p^2$. Then we average over orientations of p. It is clear that if there is only one vector q then the averages are given by the formula (34) in which one p^2 is replaced by pq. Some less trivial averages are

$$\begin{aligned}
\overline{p_\alpha p_\beta q_\mu q_\nu} &= \frac{1}{(D-1)D(D+2)} \Big[((D+1)p^2q^2 - 2(pq)^2) \delta_{\alpha\beta}\delta_{\mu\nu} \\
&+ (D(pq)^2 - p^2q^2)(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu}) \Big], \\
\overline{p_\alpha p_\beta p_\gamma q_\lambda q_\mu q_\nu} &= \frac{pq}{(D-1)D(D+2)(D+4)} \\
&\times \Big[((D+1)p^2q^2 - 2(pq)^2)(\delta_{\alpha\beta}\delta_{\lambda\mu}\delta_{\gamma\nu} + \cdots) \\
&+ ((D+2)(pq)^2 - 3p^2q^2)(\delta_{\alpha\lambda}\delta_{\beta\mu}\delta_{\gamma\nu} + \cdots) \Big], \qquad (35)
\end{aligned}$$

Figure 4: One-loop diagram for a heavy-quark condensate

$$\overline{p_\alpha p_\beta p_\gamma p_\delta q_\mu q_\nu} = \frac{p^2}{(D-1)D(D+2)(D+4)}$$
$$\times \Big[((D+3)p^2q^2 - 4(pq)^2)\delta_{\mu\nu}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \cdots)$$
$$+ \ (D(pq)^2 - p^2q^2)(\delta_{\alpha\beta}\delta_{\gamma\mu}\delta_{\delta\nu} + \cdots)\Big].$$

Of course, these formulae can be also obtained by writing down the most general forms with unknown coefficients and solving linear systems.

4 Heavy quarks

4.1 Heavy quark condensates

If the quark mass is large, all correlators can be expressed via gluon condensates only. This is also true for quark condensates (one-current correlators):

$$Q_k = \sum_n c_{kn}(m) G_n. \tag{36}$$

This is an expansion in $1/m$.

In the one-loop approximation (Fig. 4) quark condensates are given by the formula

$$Q_k = \langle \overline{q} O_k[D_\mu] q\rangle = -i \int \left(\frac{dp}{2\pi}\right)_D \langle \mathrm{Tr}\, O_k[-ip_\mu - iA_\mu] S(p)\rangle, \tag{37}$$

where A_μ is given by (4). We use the $\overline{\mathrm{MS}}$ regularization: the space dimension is $D = 4 - 2\varepsilon$,

$$\left(\frac{dp}{2\pi}\right)_D = \left(\frac{\mu^2 e^\gamma}{4\pi}\right)^{2\varepsilon} \frac{d^D p}{(2\pi)^D}, \tag{38}$$

μ is the normalization point, γ is the Euler's constant. We substitute the quark propagator (4), (5) into (37) and average the integrand over p directions in the D-dimensional space. The result is expressed via the integrals

$$I_n = \int \left(\frac{dp}{2\pi}\right)_D \frac{1}{(p^2 - m^2)^n} = \frac{i(-1)^n (m^2)^{2-n}}{(2\pi)^2 (n-1)!} \left(\frac{m^2}{\mu^2}\right)^{-\varepsilon} e^{\gamma\varepsilon}\Gamma(n-2+\varepsilon),$$
$$I_1 = \frac{im^2}{(4\pi)^2}\left(\frac{1}{\varepsilon} - \log\frac{m^2}{\mu^2} + 1\right), \quad I_2 = \frac{i}{(4\pi)^2}\left(\frac{1}{\varepsilon} - \log\frac{m^2}{\mu^2}\right),$$
$$I_n = \frac{i(-1)^n}{(4\pi)^2 (n-1)(n-2)(m^2)^{n-2}}, \quad n > 2. \tag{39}$$

Coefficients c_{kn} with $d_n \le d_k$ contain ultraviolet divergencies:

$$c_{kn}^{\text{bare}} = m^{d_k - d_n}\left[\gamma_{kn}\left(\frac{1}{\varepsilon} - \log\frac{m^2}{\mu^2}\right) + c'_{kn}\right]. \tag{40}$$

We define the quark condensate $Q_k(\mu)$ renormalized at the point μ as the sum of the series (36) from which $1/\varepsilon$ poles are removed.

We obtain the following results [7, 5]

$$Q^3 = -\frac{1}{24\pi^2}\Big[-6m^3(L+1)N + \frac{1}{m}G^4 - \frac{1}{15m^3}(G_1^6 - 6G_2^6)$$
$$+ \frac{1}{210m^5}(19G_1^8 + 16G_2^8 - 20G_3^8 - 78G_4^8 + 84G_5^8 - 4G_6^8 - 18G_7^8)$$
$$+ \cdots\Big],$$
$$Q^5 = -\frac{1}{12\pi^2}\Big[3mLG^4 - \frac{1}{m}(G_1^6 - G_2^6)$$
$$+ \frac{1}{10m^3}(3G_1^8 + 2G_2^8 - G_3^8 - 9G_4^8 + 9G_5^8 - \frac{5}{4}G_6^8 - G_7^8) + \cdots\Big],$$
$$Q^6 = -\frac{1}{12\pi^2}\Big[LG_2^6 + \frac{1}{20m^2}(18G_5^8 - \frac{1}{2}G_6^8 - 4G_7^8) + \cdots\Big],$$
$$Q_1^7 = -\frac{1}{12\pi^2}\Big[-3m^3(L+1)G^4 + \frac{1}{2m}G_1^8 + \cdots\Big],$$
$$Q_2^7 = \frac{1}{16\pi^2 m}(G_1^8 + G_2^8 - 4G_4^8) + \cdots,$$

$$Q_3^7 = \frac{1}{4\pi^2}\left[-mLG_1^6 + \frac{1}{6m}(G_3^8 - G_4^8 + 2G_5^8 - \frac{1}{2}G_6^8) + \cdots\right], \tag{41}$$

$$Q_4^7 = -\frac{1}{4\pi^2}\left[mLG_2^6 + \frac{1}{6m}(4G_5^8 - \frac{1}{2}G_6^8 - G_7^8) + \cdots\right],$$

$$Q_1^8 = -\frac{1}{12\pi^2}\left[3m^4(L + \frac{3}{2})G^4 + L(G_1^8 - 2G_3^8 - 2G_4^8) + \cdots\right],$$

$$Q_2^8 = \frac{1}{24\pi^2}\left[12m^2(L+1)G_1^6 - L(G_1^8 - G_2^8 - 2G_3^8 + 2G_4^8 - 4G_5^8 + G_6^8) + \cdots\right],$$

$$Q_3^8 = -\frac{L}{12\pi^2}G_6^8 + \cdots,$$

$$Q_5^8 = \frac{L}{6\pi^2}G_5^8 + \cdots,$$

$$Q_6^8 = 0 + \cdots,$$

where $L = \log\frac{\mu^2}{m^2}$. As we have already mentioned, the series (36) for the anomalous condensate A includes only $d = 8$ gluon condensates:

$$A = -\frac{1}{32\pi^2}(G_1^8 + G_2^8 - 4G_4^8). \tag{42}$$

Many of these results can be obtained using various physical considerations [7] instead of the described "brute force" method.

4.2 Heavy quark currents' correlators

Correlators of heavy quark currents can be expressed via gluon condensates:

$$\Pi(p) = \sum_n a_n(p^2, m)G_n. \tag{43}$$

The one-loop diagram (Fig. 5) contains two propagators of the type (4), (5) (don't forget that one of them has the vacuum momenta sink on the other side, and is given by the formulae mirror symmetric to (4), (5)!). After differentiations we obtain a formula with the integrals

$$\int \left(\frac{dk}{2\pi}\right)_D \frac{P(k)}{(k^2 - m^2)^n((k-p)^2 - m^2)^m}. \tag{44}$$

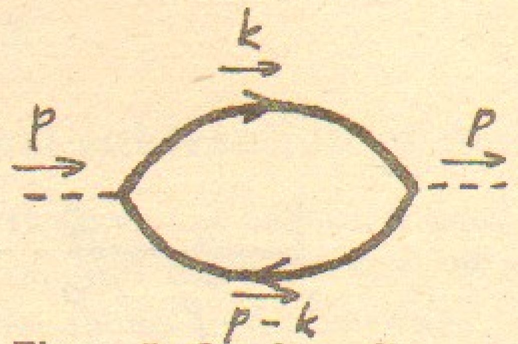

Figure 5: One-loop diagram for a correlator

The denominators can be combined using the Feynman's formula

$$\frac{1}{a^n b^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}\int_0^1 \frac{x^{n-1}(1-x)^{m-1}dx}{[xa + (1-x)b]^{n+m}}. \tag{45}$$

After the shift of the integration momentum $k \to k + xp$ the denominator becomes $k^2 + x(1-x)p^2 - m^2$; we may average the numerator $P(k+xp)$ over k directions. The integrals reduce to the form (39) with $m^2 \to m^2 - x(1-x)p^2$. We are left with one-dimensional integrals over the Feynman parameter x of the form [2]

$$J_n(\xi) = \int_0^1 \frac{dx}{[1 + \xi x(1-x)]^n} \tag{46}$$

$$= \frac{(2n-3)!!}{(n-1)!}\left[\left(\frac{a-1}{2a}\right)^n \sqrt{a}\log\frac{\sqrt{a}+1}{\sqrt{a}-1} + \sum_{k=1}^{n-1}\frac{(k-1)!}{(2k-1)!!}\left(\frac{a-1}{2a}\right)^{n-k}\right],$$

where $\xi = -p^2/m^2$, $a = 1 + 4/\xi$. Using these formulae, we can calculate any gluon contribution to a heavy quark currents' correlator. The simplest example of $d = 4$ is considered in [1]; contributions with $d = 6$ and $d = 8$ were calculated in [2].

5 Light quarks

5.1 Limit $m \to 0$ in heavy quark correlators

Let us consider the heavy-quark correlator (43) (Fig. 5) at $p^2 \gg m^2$. We can express it via both gluon and quark condensates:

$$\Pi(p) = \Pi_G(p) + \Pi_Q(p), \tag{47}$$

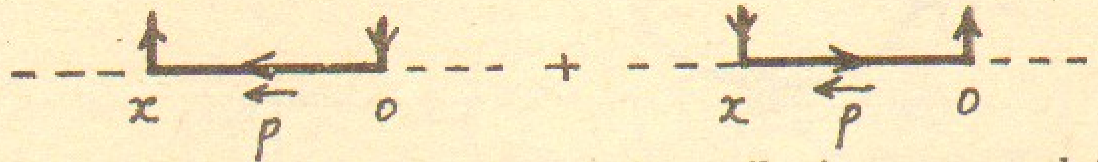

Figure 6: Quark condensates' contribution to a correlator

$$\Pi_G(p) = \sum_n a_n'(p^2,m) G_n, \quad \Pi_Q(p) = \sum_k b_k(p^2,m) Q_k.$$

Here $\Pi_G(p)$ corresponds to the contribution of the region where the virtualities of both quark and antiquark in Fig. 5 are large ($k^2 \sim p^2$), and $\Pi_Q(p)$ corresponds to the contribution of the regions where either the quark or the antiquark has a small virtuality ($k^2 \sim m^2$). These contributions are usually depicted as the diagrams of Fig. 6.

The quark condensates' contribution (Fig. 6) in the coordinate space is

$$\Pi_Q(x) = \langle \bar{q}(x) \Gamma S(x,0) \Gamma q(0) \rangle + \text{c.c.}, \tag{48}$$

where Γ is a γ matrix, and c.c. means charge conjugate. In the momentum space

$$\Pi_Q(p) = \left\langle \bar{q}(1 - i\overleftarrow{D}_\alpha \partial_\alpha - \frac{1}{2!}\overleftarrow{D}_\alpha \overleftarrow{D}_\beta \partial_\alpha \partial_\beta + \cdots) \Gamma S(p) \Gamma q \right\rangle + \text{c.c.} \tag{49}$$

We substitute $S(p)$ (4), (5) and average over p directions, and finally reduce (49) to the basis quark condensates following the Section 3.1. The contribution of A is omitted because this condensate is a gluon one. The quark contributions with $d = 7$, $d = 8$ were obtained in [6, 5].

Having obtained $\Pi_Q(p)$, we can find $\Pi_G(p)$ from the heavy-quark correlator $\Pi(p)$ using the expansions (41):

$$\begin{aligned} \Pi_G(p) &= \sum_n a_n'(p^2,m) G_n = \Pi(p)|_G - \Pi_Q(p)|_G \\ &= \sum_n a_n(p^2,m) G_n - \sum_{k,n} b_k(p^2,m) c_{kn}(m) G_n. \end{aligned} \tag{50}$$

The coefficients $a_n(p^2,m)$ have singularities at $m \to 0$ arising from the regions $k^2 \sim m^2$. These singularities have to cancel in (50) giving $a_n'(p^2,m)$ finite

at $m \to 0$. Using the results of [2], the gluon contribution with $d = 8$ was obtained in [7, 5].

5.2 Minimal subtraction of mass singularities

The method of the previous Section is good when the heavy quark correlator is already known. But when we want to calculate a light quark correlator from scratch there should be a simper way then to solve a more difficult heavy quark problem first. Such a method was proposed in [4], and used in [5] for $d = 8$ calculations.

Now we want to go to the limit $m \to 0$ before $D \to 4$. The difference of a renormalized quark condensate and a bare one is (see (40))

$$Q_k - Q_k^{\text{bare}} = -\frac{1}{\varepsilon} \sum_{d_n \le d_k} m^{d_k - d_n} \gamma_{kn} G_n \to -\frac{1}{\varepsilon} \sum_{d_n = d_k} \gamma_{kn} G_n. \tag{51}$$

At $m = 0$ all loop integrals for Q_k^{bare} vanish because they contain no scale (ultraviolet and infrared divergencies cancel each other). Therefore we obtain for the gluon contribution to a correlator (see (50))

$$\Pi_G(p) = \Pi(p)|_G + \frac{1}{\varepsilon} \sum_{d_n = d_k} b_k(p) \gamma_{kn} G_n. \tag{52}$$

Here $\Pi(p)|_G$ is calculated in the $\overline{\text{MS}}$ scheme with $m = 0$, and γ_{kn} are mixing coefficients of quark condensates Q_k with gluon condensates G_n of the same dimension (coefficients at L in (41)). Omitting $1/\varepsilon$ poles we finally obtain

$$\Pi_G(p) = \Pi(p)|_G + \sum_{d_n = d_k} \frac{db_k}{d\varepsilon} \gamma_{kn} G_n. \tag{53}$$

The quark condensates' coefficient functions b_k should be calculated at $m = 0$ up to linear terms in ε (using D-dimensional tensor and γ matrix algebra, and in particular D-dimensional averaging, see thr Section 3.3).

Acknoledgements. I am grateful to Yu. F. Pinelis for the collaboration in writing the papers [6, 7], and to D. J. Broadhurst for useful discussions of the works [3, 4, 5].

References

[1] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov. Fortschr. Phys. 32 (1984) 585.

[2] S. N. Nikolaev, A. V. Radyushkin. Yad. Phys. 39 (1984) 147.

[3] S. C. Generalis, D. J. Broadhurst. Phys. Lett. B139 (1984) 85.

[4] D. J. Broadhurst, S. C. Generalis. Phys. Lett. B142 (1984) 75.

[5] D. J. Broadhurst, S. C. Generalis. Phys. Lett. B165 (1985) 175.

[6] A. G. Grozin, Yu. F. Pinelis. Phys. Lett. B166 (1986) 429.

[7] A. G. Grozin, Yu. F. Pinelis. Zeit. Phys. C33 (1987) 419.

[8] M. J. Dugan, B. Grinstein. Phys. Lett. B256 (1991) 239.

A.G. Grozin

Methods of Calculation of Higher Power Corrections in QCD

А.Г. Грозин

Методы вычисления высших степенных поправок в КХД

Ответственный за выпуск С.Г. Попов

Работа поступила 7 февраля 1992 г.
Подписано в печать 13.02 1992 г.
Формат бумаги 60×90 1/16 Объем 1,7 печ.л., 1,4 уч.-изд.л.
Тираж 180 экз. Бесплатно. Заказ N 11

Обработано на IBM PC и отпечатано на
ротапринте ИЯФ СО РАН,
Новосибирск, 630090, пр. академика Лаврентьева, 11.