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NONLINEAR RESONANCE IN QUANTUM SYSTEMS.

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Abstract.

In the theory of the n-dimensional quantum systems the essential role is played by the resonant phenomena. The simplest problem of the nonlinear oscillator with the periodic excited force is studied in which the perturbation theory and the quasiclassical method for the calculation of the quasienergy spectrum and wavefunctions is developed. The results are applied to spectra of systems with several degrees of freedom. The long time stability of the motion is discussed and it is shown that the stochastic layers and Arnold diffusion exist only if some strong condition of quantum origin is satisfied. Some physical examples are considered.

1. Introduction.

In this work some problems of the theory of the n-dimensional quantum system are discussed. In contrast to the simple cases of the onedimensional motion and that of separable variables, still there is no even qualitative understanding of phenomena in such systems, their spectra, wave functions etc.

That the problem is very nontrivial is known from classical mechanics, under certain conditions motion of such systems changes from the stable, dynamical behaviour to the unstable, statistical one. The essential progress in understanding of these phenomena in classical systems has been made during last years and the general picture of the n-dimensional motion is somehow clarified. The progress came from the mathematical studies, in particular by A.N.Kolmogorov and V.I.Arnold /1,9/; from the theoretical and practical development of such devices as particle accelerators and magnetic "bottles"; and essentially from the "mathematical experiments" on computer with model systems (B.V.Chirikov et al /2/). The present work is the application of these ideas to the quantum case.

Since the theory in question is very little known outside the circle of specialists and still no good reviews exists, the author will try to present the main ideas so that the reader will need only some standard knowledge in mechanics like action-angle variables etc. The very brief but general picture we give from the beginning in order the reader be able to see what the content of the paper is needed for.

The main element of the theory in question is the individual resonance between the degrees of freedom of the system. Resonances are so important because they accumulate the effect of the small perturbations and with time may lead to the significant change of the motion. This fact has been realised long ago in celestial mechanics and it was shown that resonant "small denominators" in perturbation theory series lead to their divergence. But in the limited region of the phase space near the resonanse the most slowly varying or the resonant term of the Hamiltonian can be taken into account exactly. This naturally leads to the idea of separate resonance, and the account for other terms can be considered as the interaction of resonances. Let us refer the width of the region, where some resonant term is essential, as the resonance width. The behaviour of the system essentially depends on the relation between this width and the resonance separation, the so called Chirikov criterion. If the former is smaller, the motion is stable with the exception of the exponentially small unstable regions - the so called stochastic layers. In the opposite case the motion is very complicated or stochastic with the exception of the small "stability islands". Note, that the resonance width (and therefor this criterion) depend essentially on the coupling strength between the degrees of freedom of the system.

In the quantum case this two regimes also exist and in the quasiclassical limit the criterion is of course the same. The first or stable one means, that although the

coupling changes the wavefunctions significantly, they still can be divided into families of states with small overlap, which have rather simple and calculatable properties. This is the content of the present work. The stochastic regime will be discussed elswhere. Let us only say here, that in this case the decomposition of the exact states over the uncoupled ones is very complicated and in some sense "random".

The plan of the paper is the following. Chapter 2 is devoted to the problem of onedimensional motion under the influence of the external periodic force. This is the simplest problem were the nonlinear resonance is present. Chapter 3 generalizes the results to the resonance between the degrees of freedom of the conservative system. In both cases the wavefunctions and energy spectra (in the first one it is the quasienergy /3/) can be calculated. Depending on the relative value of the resonance width and the Plank constant two regims are possible, in which the perturbation theory or the quasiclassical approximation can be applied. As a simple example the quasienergy spectra are found for the linear parametric and external resonances in the Appendix B, which is already known from the exact solution /4,5/.

The interaction of the resonances is discussed in chapter 4. They lead to the appearence of the stochastic layers on which the universal instability - the Arnold diffusion - takes place. In the quantum case these phenomena exist only if some condition is satisfied, which turns out to be so severe that it is not fulfilled in some systems, at first sight completely classical.

Finally, some words about the applications. The results of the chapter 3 can be used for the study of spectra of electric and acoustical resonators and molecules. The applications of related ideas to nuclear physics are discussed in the review /13/, and to field theory in /14/. The results of the chapter 4 are of interest in cases when the long time stability of the motion is needed. We consider briefly the magnetic "bottles" and colliding beams in the text. Very interesting applications are those to atomic motion in molecules, which is discussed in separate paper. For example, the twoatomic molecule vibration in the intence resonant electromagnetic wave is just the problem of the chapter 2.

2. Separated resonance of quantum nonlinear oscillator.

In contrast to stationary problems of quantum mechanics the time dependent ones are studied relatively little for in this case there is no so powerful method as the expansion over the eigenfunctions of the Hamiltonian. But in the case of periodic time dependence there exists other set of states which in large measure substitute for them. They are the states with definite quasienergy defined by the condition /3,4/:

$$\Psi_n(t+T) = e^{-\frac{i}{\hbar}\varepsilon_n T}\,\Psi_n(t) \qquad (1)$$

Here $\Psi_n(t)$ are the wavefunction of such state, ε_n is quasienergy, T is the period of the time dependence. Being the eigenfunctions of the operator $T\exp\left(-\frac{i}{\hbar}\int_0^T \hat{H}\,dt\right)$ of the evolution over one period, they form the complete orthogonal set. Thus the general solution of the Schrodinger

equation is the superposition of these states with coefficients independent on time. Thus the knowledge of such decomposition is sufficient for understanding of the complete evolution of a given state.

For the completeness let us begin with the case of small enough external perturbation. It is clear, that in this case the states $\Psi_n(t)$ are close to unperturbed ones, as well as quasienergies ε_n to unperturbed energies. The perturbation theory for them is presented in Appendix A. The only difference with the standard case is that the initial condition is not given, but the condition (1) must be satisfied. Let us present only the second order result for quasienergy which is rather familiar:

$$\varepsilon_n = E_n + V_{nn}^{(0)} + \frac{1}{\hbar}\sum_{m,l} \frac{|V_{mn}^{(l)}|^2}{l\Omega - \omega_{mn}}\,; \quad V_{mn}^{(l)} = \oint dt\, e^{il\Omega t}\langle m|V|n\rangle \qquad (2)$$

Interesting, that this result is exact for linear resonance /5/.

In the case of the resonance the denominator is small and the perturbation theory is not valid. If only one transition between two levels is resonant the problem can be easily solved exactly, see e.g. /11/. We remind this for it is instructive to calculate quasienergy in this case. For $V_{mn} \gg \hbar|\Omega - \omega_{mn}|$ the result is $\varepsilon_{m,n} = E_{m,n} \pm |V_{mn}|$. Note that the result is now $\sim|V_{mn}|$ and not $|V_{mn}|^2$ as in (2). This is a consequence of the autophasing to be discussed below.

More interesting is the case when the number of states taking part in the transitions is not small. The discussion above shows that trasitions are effective till the frequency shift caused by nonliniarity will not reach the value of the perturbation:

$$| E_n - E_{n_0} - \hbar\Omega(n-n_0) | \lesssim V \qquad (3)$$

Here n_0 correspond to the resonance point $\frac{dE}{dn}(n_0) = \Omega$
Expanding E_n near n_0 one has

$$E_n = E_{n_0} + \hbar\Omega(n-n_0) + \frac{1}{2}\frac{d\omega}{dn}(n-n_0)^2 + \ldots \qquad (4)$$

From (3,4) we find the resonance width

$$|n-n_0| \lesssim \left(\frac{2V}{\hbar\frac{d\omega}{dn}}\right)^{1/2} \equiv \Delta n_R \qquad (5)$$

For small values of $\Delta n_R \ll 1$ the perturbation theory can be used, for its large value

$$V \gg \hbar\frac{d\omega}{dn} \qquad (6)$$

the quasiclassical approximation is natural.

We shall assume also, that the separation between the different harmonics of the force $\mathcal{D}$ is much larger than the width (5), they are separated. This condition can be written as

$$V \ll \hbar\mathcal{D}^2 / \frac{d\omega}{dn} \qquad (7)$$

If it is so, our discussion can also be applied to unperiodic force , which contains discrete set of incommensurable harmonics. For the periodic force $\mathcal{D} = \frac{2\pi}{T}$ and for small enough nonlinearity the separated resonances exist. In more general case for high harmonics density (6) and (7) are incompatible.

The quasiclassical approximation of the resonances is rather simple because with the quasiclassical accuracy one can use the action-angle variables, in which the whole theory is more transparent. In connection with this let us note, that the canonic transformations in quantum mechanics are not yet developed. The "practical"difficulty is the noncommutation of coordinates and momenta in the generating function and the Hamiltonian. But in quasiclassical approximation the extra "counteterms" are not essential for they arise due to commutators and contain extra powers of small parameter $\hbar$. The difficulties in more accurate transition to action-angle variables in quantum case are discussed in /10/.

Begin with quantisation of the nonlinear oscillator in the absence of the perturbation which is rather trivial. The Schrodinger equation is

$$H_0(\hat{I})\,\Psi_E(\theta) = E\,\Psi_E(\theta)\ ;\ \hat{I} = i\hbar\frac{\partial}{\partial\theta} \qquad (8)$$

and its solution just

$$\Psi_E(\theta) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{i}{\hbar}I(E)\cdot\theta\right] \qquad (9)$$

where $I(\theta)$ is the function inverse to $H_0(I)$. The wavefuction is periodic in θ and therefore

$$I_n = n\hbar \quad ; \quad E_n = H_0(n\hbar) \qquad (10)$$

Clear that these formulae are valid only in quasiclassical case.

The Hamiltonian with perturbation looks as

$$H(I,\theta,t) = H_0(I) + V(I,\theta,t) \qquad (11)$$

Since the perturbation $V(I,\theta,t)$ is periodic in time and angle, it can be expanded into double Fourier series:

$$V(I,\theta,t) = \sum_{m,n} V_{mn}(I)\exp(im\theta - in\Omega t) \qquad (12)$$

Resonances correspond to such values of I at which

$$k\omega(I_0) = \ell\Omega \ ; \quad \omega(I) \equiv \frac{dH_0}{dI} \qquad (13)$$

The resonant terms in (12) are those with m, n such that $mk = n\ell$, for their time dependence is the most slow. In other terms, only part of the perturbation $U(I,\alpha)$ which depend on resonance phase only is of main interest, where $\alpha \equiv k\theta - \ell\Omega t$. In order to simplify formulae we restrict ourselves to the case $k = \ell = 1$.

Considering only the resonant terms and changing the variables from I, θ to I_α, α we obtain the resonant Hamiltonian

$$\tilde{H}(I_\alpha,\alpha) = H_0(I_\alpha) - \Omega\cdot I_\alpha + U(I_\alpha,\alpha) \qquad (14)$$

which is not time dependent.

Before we proceed to its quantisation let us show, that its eigenfunctions have definite quasienergy, coinciding with its eigenvalue. To see this it is sufficient to return to initial variables

$$\Psi_n(\theta,t) = \tilde{\varphi}_n(\theta - \Omega t)\cdot\exp(-\tfrac{i}{\hbar}\tilde{E}_n t) \qquad (15)$$

where $\tilde{\varphi}_n(\alpha)$ is the eigenvalue of $\tilde{H}(I_\alpha,\alpha)$. Since it is periodic in α they have the property (1) characteristic for states with definite quasienergy. Let us remind that the quasienergy is defined up to addition of $\hbar\Omega$ times some integer number, just like quasimomentum.

The discussion of the nonlinear resonance often is made for the case of very small nonlinearity in hope for some simplification. But in reality the situation is inverse /2/, for large enough nonlinearity the problem is more simple and, what is even more important, it becomes universal. In this case $\Delta n_R \ll n_0$ or in other terms

$$\frac{V}{E(n_0)} \ll \frac{n_0}{\omega}\frac{d\omega}{dn} \qquad (16)$$

so that the resonant Hamiltonian (14) can be expanded near $n_0 = \hbar I_\alpha^{(0)}$ and reduced to such universal one /2/

$$\tilde{H}(I_\alpha,\alpha) \approx \frac{1}{2\hbar}\frac{d\omega}{dn}(I_0)\cdot(I_\alpha - I_\alpha^{(0)})^2 + U(I_\alpha^{(0)},\alpha) \qquad (17)$$

analogous to that for particle with mass $\hbar / \frac{d\omega}{dn}$ in the potential well $U(I_\alpha^{(0)},\alpha)$. The regions of limited and unlimited phase motion are separated by the separatrissa (see Fig.1). The existence of limited motion of the phase or phase oscillations is the famouse Veksler autophasing /12/ of the nonlinear oscillator according to the force. From (17) it is clear that the separatrissa separate the region in action with the width

$$\Delta I_R = \left(\frac{2\hbar \, \Delta U}{d\omega/dn} \right)^{1/2} ; \quad \Delta U = \max U - \min U \qquad (18)$$

which of course only by Plank constant differs from (5).

The quasiclasiical quantisation of this Hamiltonian is made in a standard way. In the region of infinite motion of the phase (we shall say - outside of the resonance) the wavefunctions are

$$\begin{aligned} \tilde{\varphi}_n(\alpha) &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{i}{\hbar} \int^{\alpha} I_{ef}(\alpha')\, d\alpha' \right] \\ I_{ef}(\alpha) &= I_\alpha^{(0)} \pm \left[\frac{2\hbar}{d\omega/dn} \left(\tilde{E}_n - U(I_\alpha^{(0)}, \alpha) \right) \right]^{1/2} \end{aligned} \qquad (19)$$

and the quantisation condition with our accuracy

$$\int_0^{2\pi} I_{ef}(\alpha)\, d\alpha = 2\pi n \hbar \qquad (20)$$

Inside the resonance solutions (19) are not independent and coincide in the turning points. The solution is as usial such their combination which decreases into classically inaccessible region. Far enough from separatrissa the quantisation condition is: ($\alpha_{1,2}$ are turning points)

$$\int_{\alpha_1}^{\alpha_2} d\alpha \left[\frac{2\hbar}{d\omega/dn} \left(\tilde{E}_n - U(I_\alpha^{(0)}, \alpha) \right) \right]^{1/2} = \pi n \hbar \qquad (21)$$

Note that here there is no factor 2 as in (20), but any solution is the superposition of both ones (19), so the total number of states remains the same.

If the condition (16) is not fulfilled then different types of resonances exist: external, parametric etc. In principle they are quantized in the same way, but for Hamiltonians of different types. The examples of such method are given in Appendix B. The region near separatrissa will be discussed

below. The expansion of the found states with definite quasienergy over stationary states of unperturbed oscillator is made in Appendix C. The results support our main conclusion that the essential change in wavefunction takes place only inside the resonance with the strong decrease outside. Note that in contrast to the unperturbed states, the exact ones have nonzero average phase which is rotated with the frequency of the force. That is why the quasienergies are first order in perturbation.

3. Resonances in autonomic systems.

Begin with the most simple case of the systems with several degrees of freedom, namely those in which variables can be separated and the set of independent oscillation modes (nonlinear by themselves) exist. For simplicity let us consider the two dimensional system which total energy is

$$E(n_1, n_2) = E_1(n_1) + E_2(n_2) \qquad (22)$$

Here n_1, n_2 are the quantum number of oscillators and E_1, E_2 are their energies. Even this system has inhomogeneous spectrum because of the resonances. Indeed near the points where

$$K\omega_1(n_1) = \ell\, \omega_2(n_2) \qquad (23)$$

(K, ℓ integer) some clustering of the levels appears, for the change of $K \cdot N$ quanta of the first mode to $\ell \cdot N$ quanta of the second one results in the following change of the total energy:

$$E(n_1 + KN, n_2 - \ell N) - E(n_1, n_2) = \frac{1}{2}(\omega_1' K^2 + \omega_2' \ell^2) \cdot N^2 \qquad (24)$$

Here $\omega'_{1,2}$ are the nonlinearities of the modes. If they are small compared to $\omega_{1,2}$ (frequences do not change significantly from level to level) the quantity (24) is also small. Such example in molecular physics is known as the Fermi resonance.

From (24) one can see, that the level density near resonance behaves as $\varrho(E) \sim (E-E_0)^{-1/2}$, and there is the succession of such thres holds due to a given resonance, separated by distance $k\omega_1 = \ell\omega_2 = \Omega$. The spectrum of the discussing system is schematically shown in Fig.2a.

Now let us turn in some small per turbation, causing the coupling between modes. and having matrix elements of the order of V . It is clear that in first order it will express itself in places of the level clustering, mixing

$$\Delta n_R \sim \left(\frac{V}{\omega_1' k^2 + \omega_2' \ell^2} \right)^{1/2} \qquad (25)$$

levels. For small $\Delta n_R \ll 1$ perturbation theory can be used, for $\Delta n_R \gg 1$ the quasiclassical one. For the case of separated resonances everything can be done explicitly similar to the problem of the preceeding chapter. The main poins are the account for the resonant part of the perturbation only and the transition from variables $I_1, I_2, \theta_1, \theta_2$ of separate modes to

$$\alpha = k\theta_1 - \ell\theta_2 \qquad \theta = \tfrac{1}{2}(k\theta_1 + \ell\theta_2)$$
$$I_\alpha = \frac{I_1}{2k} - \frac{I_2}{2\ell} \qquad I_\theta = \frac{I_1}{k} + \frac{I_2}{\ell} \qquad (26)$$

The result is the resonant Hamiltonian

$$\tilde{H}(\alpha, I_\alpha I_\theta) = \Omega(I_\theta - I_\theta^{(0)}) + (\omega_1' k^2 + \omega_2' \ell^2)(I_\alpha - I_\alpha^{(0)})^2 + U(\alpha, I_\alpha^{(0)}, I_\theta^{(0)}) \quad (27)$$

where $\Omega = \Omega(I_\alpha^{(0)}, I_\theta^{(0)})$ is the frequency in the resonance. This expression corresponds to the approximation (16) like (17). But in any case in resonant approximation I_θ is conserved and I_α, α oscillate. We may call the discussing phenomenon the mutual phasing of the modes. The spectrum of (27) is schematically shown in Fig.2b, the detailed structure of clusters depends on the particular form of $U(\alpha)$ and is easily calculatable. Let us note only, that since I_α, I_θ are integer in units $\hbar$, then I_1, I_2 are divisible by k, ℓ . It means that for a given resonance only some "sublattice" of states is important.

The generalization to manydimensional case is done similarly. The general resonance

$$\sum_{i=1}^{N} k_i \cdot \omega_i(n_i^{(0)}) = 0 \qquad (28)$$

picks out the surface in the phase space. Their interseptions are more strong resonances up to periodic one $k_1\omega_1 = k_2\omega_2 = \ldots = k_N\omega_N$, lying on oneparameter line. In this cases several resonant phase appears and although our method reduce the number of variables, the problem in generalcase remains complicated. and further simplifications are needed.

In the works /6-8/ the quasiclassical approximation for Green functions is studied in the path integral method.

The applications of their results in nuclear physics are reviewed in /13/ and in field theory in /14/. They consider only orbits close to periodic ones. In /8/ the spectrum near the "threshholds" are found to be

$$E_{m,n} = E_{m,n}^{(cl)} + \sum_{i=1}^{N-1} (n_i + \tfrac{1}{2}) \hbar \beta_i \qquad (29)$$

where $E_{m,n}^{(cl)}$ correspond to Bohr-Zommerfield quantization along the orbit, and $\beta_i = \frac{d\nu_i}{dT}$ where ν_i are the so called stability angles. In our terminology β_i are the frequencies of the phase oscillation and (29) is evident consequence of (27) for small amplitude of these oscillation. So our approach gives this result in much more simple way, because we first make transformation to convenient variables and only then make the quantisation. Moreover, the smallness of the phase oscillation is not needed but only the validity of the quasiclassics, which is evident for the twodimensional case discussed above.

4. Interaction of resonances and stability of the motion.

With the account of some nonresonant term $W(I,\alpha,t)$ the Hamiltonian becomes

$$\widetilde{H}(I_\alpha, \alpha, t) = \widehat{H}(I_\alpha, \alpha) + W(I_\alpha, \alpha, t) \qquad (30).$$

where the first term is the resonant Hamiltonian (14). Note that we practically return to the original problem of the nonlinear oscillator (this time that of phase oscillations) with periodic perturbation.

The question we are going to discuss is whether the effect of the perturbation is small. In this point it is convenient to introduce some dimensionalless parameter ε representing the strength of the coupling between modes. The nonresonant term W in (30) is of the order of ε as well as the resonant one already taken into account, but still the effect of W inside the resonance is small together with ε. The reason is that the frequency of W is the difference between our resonance and of the other one, interacting with it. So this frequency Ω^* is independent of ε while the frequency of phase oscillation $\Omega_f \sim \sqrt{\varepsilon}$. Therefore for small ε $\Omega^* \gg \Omega_f$. Note that it is just the condition for resonances to be separated. As the consequence of this, the matrix element of W is "adiabatically small" /2/ because of the factor

$$\langle W \rangle \sim \exp\left(- C_1 \frac{\Omega^*}{\Omega_f}\right) \sim \exp\left(- C_2 / \sqrt{\varepsilon}\right) \qquad (31)$$

Here C_1, C_2 are some constants. Note that this result is essentially nonanalitical in ε, and the perturbation expansion in ε diverges.

Since the effective perturbation is small its effect can be significant only due to resonances. They are called the second order resonances /15,2/ and are of the type

$$\Omega_f = \frac{\Omega^*}{n} \; ; \; (n \text{ integer}) \qquad (32)$$

The particular role is played by the vicinity of the separatrissa where Ω_f goes to zero and resonances (32) exist for any given Ω^* . Moreover, they form the convergent set of such resonances and for any amplitude of the perturbation there is some overlap of their widths. These regions are called the stochastic layers /2/ and their relative measure is of the order of (31).

Although for small ε or for separated resonances the unstable or stochastic region is very small, its existence in any classical nonlinear system (of course except the integrable ones) is essential. For one oscillator it is not so essential for this region is "squeesed" between the stable ones /1/, but for manydimensional systems it has the form of the intersepting net of thin layers and the motion in it can lead to qualitative change of the motion. The example of this kind has been first constructed by Arnold /9/ and the phenomenon is known as the Arnold diffusion /2/. These phenomena becomes practically important in the problems when the long time stability of the motion is needed.

With this introduction let us turn now to quantum case we are interested in. Because of the discreteness of quantum energy levels the phase oscillation frequency Ω_f does not reach zero but has the smallest value of the order of

$$\Omega_f^{(min)} \sim \Omega_f^{(typical)} / \ln(\Delta n_R) \qquad (33)$$

This is clear from the quasiclassical quantisation near the maxima of the potential. From this it is clear that the quasiclassical theory of the second order resonances can be va-

lid if there are many levels near separatrissa or $\ln(\Delta n_R) \gg$ $\gg 1$, which is much more strong condition than that of quasiclassical condition $\Delta n_R \gg 1$ for the resonance itself.

Besides that the perturbation must be large enough for the formation of the second order resonances. The condition is the same as (6). Otherwise the perturbation makes no significant changes in wavefunctions and can be taken into account by the perturbation theory. This means, that the instability always present in classical case, disappears under certain condition in quantum theory. The explicit form of this condition can be obtained from (31) and rewritten as a condition on ε

$$\varepsilon > \left[\frac{c_2}{\ln(I_0/\hbar)} \right]^2 \qquad (34)$$

where I_0 is some characteristic value of action in the considered problem. Since it enter only under log dependence, there is no need to make it more accurate. Note, that this condition is very strong and may not hold for very large systems.

Let us briefly mention some examples in which the discussed phenomena can be important. The first one is the magnetic "bottles" in which electrons are confined in longitudinal direction by the magnetic "corks" and in the trans verse one by the magnetic field /16/. The stability of the motion is studied in details in /2/. The usual adiabatic invariant conservation is violated by the resonances between the Larmor frequency $\omega_H = \frac{eH}{mc}$ and the longitudinal motion one $\Omega_\parallel$. The effective perturbation is then

$$\mathcal{E} \sim \exp\left(-c_3 \frac{\omega_H}{\Omega_\parallel}\right), \quad c_3 = O(1) \qquad (35)$$

and the condition for the existence of the stochastic layers and the particle loss due to Arnold diffusion (34) becomes

$$\frac{\Omega_\parallel}{\omega_H} > \frac{c_3}{2\ln\ln(I_c/\hbar)} \qquad (36)$$

Estimates for the conditions of the experiments /17/ shows that the r.h.s. of (36) is of the order of 0.1 and the l.h.s. 0.12-0.2. It means that the boundary of quantum region was very close.

Another example, also taken from /2/, is the stability of particles in storage ring. The transeverse oscillation of particles have resonances with each other, and the perturbation is caused by the colliding beams, inhomogeneous ionisation along the orbit etc. In this example nonlinearity is of the order of $\mathcal{E}$ itself and $\Omega_f \sim \mathcal{E}$. Therefore (34) must read without the second power in the r.h.s. The author is indebted to B.V.Chirikov for this comment.

Experiments /18/ have revealed the Arnold diffusion at $\mathcal{E} \sim 1/20$. The estimate according to our quantum condition gives $\mathcal{E} \gtrsim 1/30$. At ISR CERN some slow beam widening is observed, which has been connected /19/ also with the Arnold diffusion, but according to our estimate $\mathcal{E}$ in this case is at least several times smaller than that given by our condition. So the question remains open.

The last example, which has no practical meaning but still is rather interesting and instructive, is the Solar system. The question is whether one planet can leave the system. This question has been "solved" many times in the sense that more and more weak instabilities were studied. Now it is known that Kolmogorov stability is violated only by effects like the Arnold diffusion. This makes the problem for planets purely academic for the instability time is much larger than the age of the Universe. But for asteroids these effects can be more effective and they can explain the so called Kirkwood lukes in the resonances with Jupiter /2/.

But it is still interesting, that the parameter $\mathcal{E}$ for planets even does not satisfy the condition (34) which demands it to be larger than something like 10^{-5}. The main smallness of $\mathcal{E}$ comes from the evident factor $M/M_\odot$ (M is the planet mass, $M_\odot$ is that of the Sun) and the power of excentricitet e

$$\mathcal{E}_{n_1, n_2} \sim \frac{M}{M_\odot} \cdot e^q \; ; \quad q = |n_1 - n_2| \qquad (37)$$

where n_1, n_2 from the considered resonance $\omega_1 n_1 = \omega_2 n_2$. For any planets it is smaller than 10^{-5}, although for resonant asteroids the condition is fulfilled.

In the end of the work let us discuss briefly what happens when two resonances becomes closer. The amplitude of the resonances of the second and further orders increase rapidly according to (31) and the stochastic region also increases. In quantum language the exact states become more and more complicated till they contain in equal part any of the state from the internal regions of two resonances in question. When all resonances overlap they probably contain all the unperturbed states in the energy region of the order of coupling V . If this hypothesis is true, it will explain the microcanonical distribution - the bases of the statistical mechanics.

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This work will be absolutely impossible without the works of B.V.Chirikov and many fruitful discussions with him.

Appendix A.

Perturbation theory for quasienergy.

Let us expand the state we look for over the unperturbed ones Ψ_k with energies E_k

$$\varphi_n(t) = \sum_k a_{nk}(t)\,\Psi_k \tag{A1}$$

The definition (1) for ε_n, φ_n is

$$a_{kn}(t+T) = e^{-\frac{i}{\hbar}\varepsilon_n T} a_{kn}(t) \tag{A2}$$

and the Schrodinger equation is as usual

$$i\hbar \dot{a}_{kn}(t) = E_k a_{kn}(t) + \sum_m V_{km}(t) a_{mn}(t) \tag{A3}$$

In the zeroth order $\varepsilon_k^{(0)} = \delta_{kn} E_n \; ; \; a_{kn}^{(0)}(t) = \delta_{kn} \exp(-\frac{i}{\hbar}E_n t)$

The first order solution of (A2) is

$$a_{kn}^{(1)}(T) = \exp(-\tfrac{i}{\hbar}E_k T)\, a_{kn}^{(1)}(0) - \exp(-\tfrac{i}{\hbar}E_k T)(e^{i\omega_{kn}T} - 1)\sum_L \frac{V_{kn}^{(L)}}{\omega_{kn} + L\Omega} \tag{A4}$$

Here L is the number of the harmonics of perturbation

$$V_{kn}^{(L)} = \oint dt\, \exp(iL\Omega t) \langle k | V(t) | n \rangle \tag{A5}$$

For $k = n$ one has $\varepsilon_n^{(1)} = V_{nn}^{(0)}$ and for $k \neq n$ divide over the T dependent factor one has

$$a_{kn}^{(1)}(t) = e^{-iE_n t} \sum_L \frac{V_{kn}^{(L)} e^{iL\Omega t}}{\omega_{kn} - L\Omega} \tag{A6}$$

The quantity a_{nn} , as usual, is of the second order. Proceeding in the same way in the second order one finds

$$\varepsilon_n^{(2)} = \sum_{m,L} \frac{1}{L\Omega - \omega_{mn}} |V_{mn}^{(L)}|^2$$

$$a_{kn}^{(2)}(t) = e^{-iE_n t}\left[\frac{iV_{nn}^{(0)}T}{e^{i\omega_{kn}T} - 1}\sum_L \frac{V_{kn}^{(L)}}{L\Omega - \omega_{kn}} + \sum_{L,L',m} \frac{V_{km}^{(L')} V_{mn}^{(L)}}{(L'\Omega - L\Omega - \omega_{km})(L\Omega - \omega_{mn})}\right. \tag{A7}$$

Appendix B. Examples of quantization of resonances.

The resonance Hamiltonian of the general form (14) with arbitrary dependence on action and angle is not the operator with known properties and it is unknown how to find its eigenfunction. However in the quasiclassical approximation we are only dealing with they in any case are of the type

$$\Psi(\alpha) \propto \exp\left(-\frac{i}{\hbar}\int I_{ef}(\alpha')\, d\alpha'\right) \tag{B1}$$

where $I_{ef}(\alpha')$ is the solution of the equation $\tilde{E} = \hat{H}(I_{ef},\alpha')$. Clear that in the classically accessible region only real solutions are needed. In principle there may be many of them crossing in points α_i, the generalization of the usual turning points. In any such crossing of two solutions they acquire the imaginary part and since only decreasing solution is needed the relative phase of the combination of these solutions, entering into true wavefunction, can be found. The quantization condition (21) is generalized as

$$\int_{\alpha_1}^{\alpha_2} d\alpha' \left[I_{ef}^{(1)}(\alpha') - I_{ef}^{(2)}(\alpha') \right] = 2\pi n \hbar \qquad \text{(B2)}$$

Let us present some simple examples of the use of this method. Begin with the linear $\left(\frac{d\omega}{dn} = 0\right)$ parametric resonance or frequency modulation of the linear oscillator. Ommitting simple calculations we present the resonance Hamiltonian (14) for this problem:

$$\hat{H}(I_\alpha,\alpha) = \Delta\omega \cdot I_\alpha + V(\alpha)\cdot I_\alpha \qquad \text{(B3)}$$

where $\Delta\omega = \omega_0 - \Omega/2$ is the frequency shift of the oscillator and the force. The equation for $I_{ef}(\alpha)$ is trivial and since the wavefunctions must be periodic we have the quasienergy spectrum

$$\hat{E}_n = n\hbar\lambda; \quad \lambda^{-1} = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\alpha}{\Delta\omega + V(\alpha)} \qquad \text{(B4)}$$

in agreement with the exact solution /5,4/. The condition $\Delta\omega > V(\alpha)$ coincides with the well known condition of the stability of the parametric resonance. This fact is gene-

ral, the stable resonances have discrete quasienergy spectrum and the unstable ones, leading to infinite growth of amplitude, have continuous spectra.

The next simple example is the linear resonance with usual exciting force, which has the resonant Hamiltonian

$$\tilde{H} = \Delta\omega \cdot I + f \cdot I^{1/2} \cos\alpha \qquad \text{(B5)}$$

the quantity $I_{ef}(\alpha)$ is a little bit more complicated

$$I_{ef}(\alpha) = \frac{f^2\cos^2\alpha}{2\Delta\omega^2} + \frac{\tilde{E}_n}{\Delta\omega} - \frac{f\cos\alpha}{2\Delta\omega^2}\sqrt{f^2\cos^2\alpha + 4\tilde{E}_n\cdot\Delta\omega} \qquad \text{(B6)}$$

but in the quantisation the last term does not contribute and $\tilde{E}_n = \Delta\omega\cdot n + \frac{f^2}{4\Delta\omega}$ again in agreement with /5,4/. The coincidences of the quasiclassical expressions with the exact ones are known in many problems connected with the linear oscillator.

Finally more complicated example to be disscused in details elsewhere: the nonlinear resonance near the ground state, where (16) is not valid. Such is the situation in the problem of molecule vibration excitation by the intense resonant field. The resonant Hamiltonian is of the type

$$\hat{H} = \Delta\omega\cdot I + \frac{\omega' I^2}{2} + f(\alpha)\cdot I^{1/2} \qquad \text{(B7)}$$

and for $I_{ef}(\alpha)$ four solutions may exist. The I representation is more convenient here than the α one used above. For $\Delta\omega = 0$ the resonance width in this case is

$$\Delta I_R = \left(\frac{2\Delta f}{\omega'}\right)^{2/3}; \quad \Delta f = \max f - \min f \qquad \text{(B8)}$$

Appendix C. The decomposition of states with definite quasienergy in the unpertubed states.

For this purpose we need to calculate integrals

$$a_{nk} = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \cdot \exp\left[in\alpha - \frac{i}{\hbar}\int^{\alpha} I_{ef}(E_k\alpha')\,d\alpha'\right] \qquad (C1)$$

which can be done by the saddle point method. The saddle point is determined from the equation

$$n\hbar = I_{ef}(E_k\alpha_0) \qquad (C2)$$

which has the real solutions only for n close to k. In this region

$$|a_{nk}|^2 = \frac{1}{2\pi\, dI_{ef}(\alpha_0)/d\alpha} \qquad (C3)$$

otherwise α_0 is complex and the overlap is exponentially small. If n lies inside the resonance then the region where α_0 is real is just the internal part of the resonance.

To give the example, let us present the result for the universal resonance (17) with $u(\alpha)=u\cdot\cos\alpha$. For $|n-k|\gg\Delta n_R$ (that is for the state distant from the resonance) $\alpha_0 \simeq \pi + 2i\ln\frac{|n-k|}{\Delta n_R}$ has large imaginary part and

$$|a_{nk}|^2 \sim \left(\frac{|n-k|}{e\,\Delta n_R}\right)^{-4|n-k|} \qquad (C4)$$

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Figure captions.

Fig.1. The schematic dependence of the phase oscillation potential as a function of the resonance phase α . The dashed line shows the separatrissa, separating the region A of the finite motion from that B or the infinite phase rotation.

Fig.2. The spectrum of the two dimensional system without the coupling between modes (a) and in its presence (b). The conribution of only one resonance is shown. The dashed curve represent the regular part of the spectrum (the phase space approximation) . The structure of the "level clasters" are shown also in more details. In the case (a) $\rho(E) \sim (E-E_c)^{-1/2}$ near the threshhold (24), while in the case (b) it corresponds to phase oscillations (27) with ponential shown in Fig.1. Near the thres hold $\Delta\rho = 1/\hbar\Omega_f$ in agreement with (29), the further grouth is the influence of the separatriss

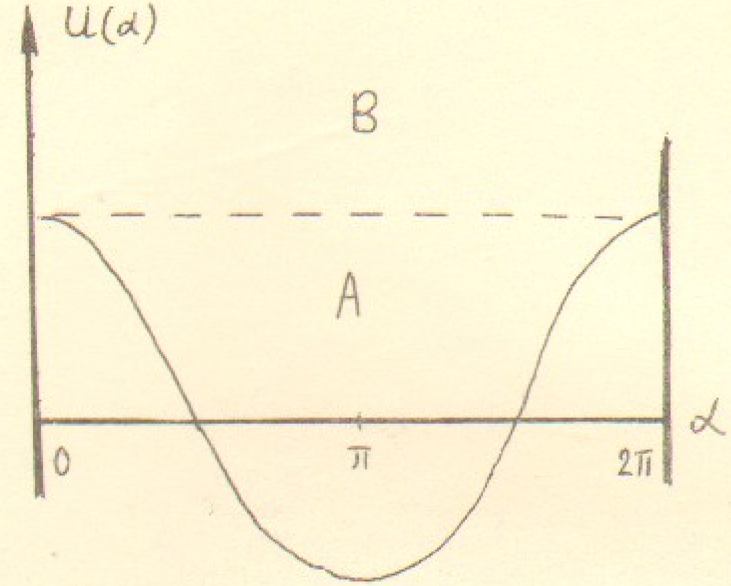

Fig. 1

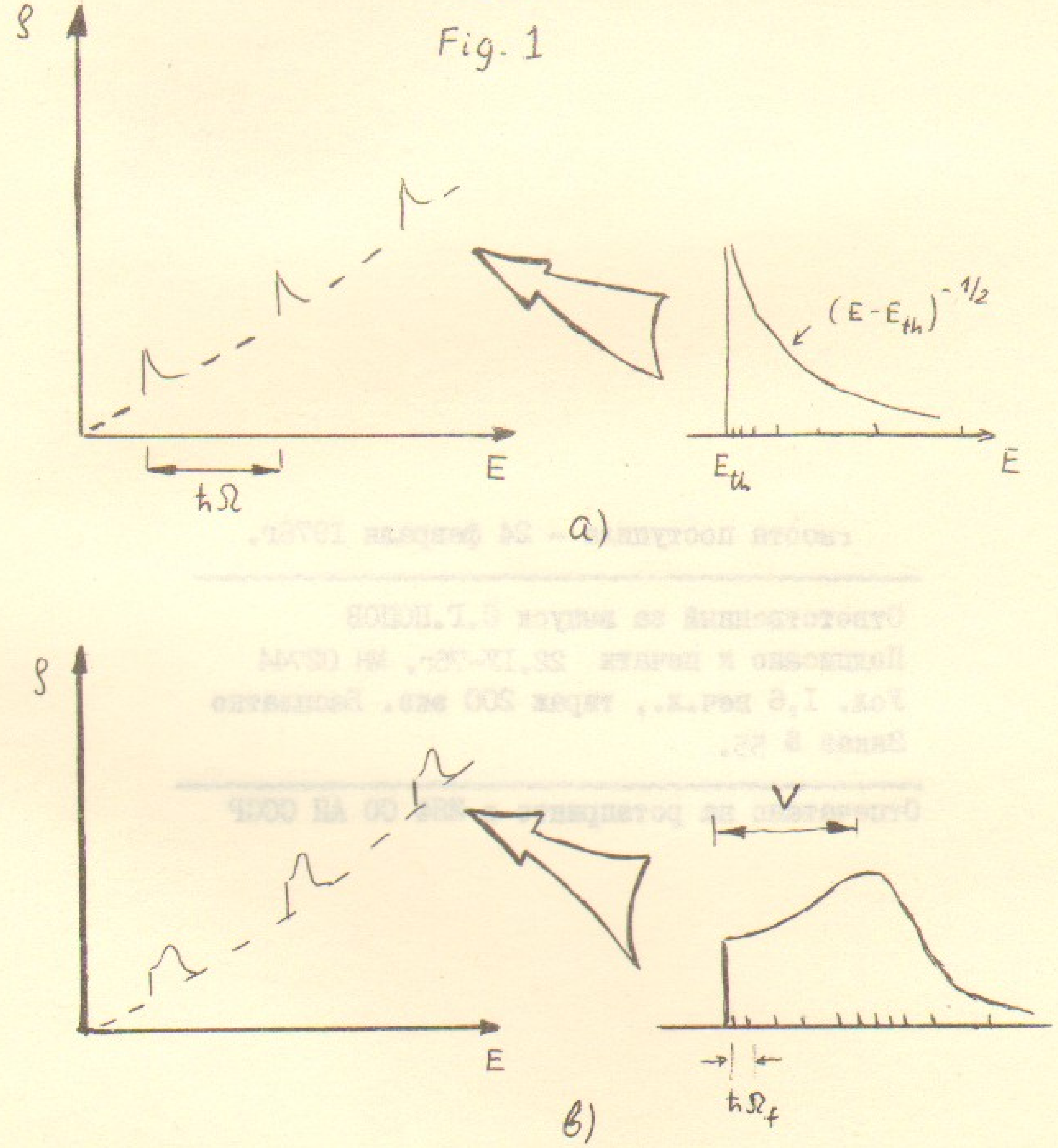

Fig. 2

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